Barankin, McAulay-Seidman and Cramér-Rao Bounds on Matrix Lie Groups

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Abstract

In this article, we first derive a general intrinsic Barankin bound (IBB) for unknown parameters lying on Lie groups (LGs), and its intrinsic McAulay-Seidman bound (IMSB) approximation. Second, the IMSB expression is used to revisit the intrinsic Cramér-Rao bound (ICRB) on LGs. Indeed, an analytic expression of the ICRB, which is a special IMSB case, is obtained from the latter. Finally, closed-form expressions for both IMSB and ICRB are obtained for Euclidean and LG observation models depending on parameters lying in $SO(3)$ and $SE(3)$. The validity of the these IMSB and ICRB expressions, with respect to the intrinsic mean square error, is shown via numerical simulations to support the discussion.

Key words: Performance bounds, McAulay-Seidman bound, Cramér-Rao bound, estimation on manifolds, Lie groups.

1 Introduction

Performance bounds are of major interest in various signal processing areas. Indeed, these bounds allow to obtain the minimum achievable estimation mean square error for an estimator that is built from a given statistical model. When the unknown parameters' vector of interest is subject to some mathematical properties, it can be constrained to lie on a smooth manifold. In the last decades, estimation of parameters lying on a manifold raised much interest, especially in the image and signal processing communities [1][2][3]. For instance, in radar target tracking [4] or texture classification [5], it is of interest to estimate unknown covariance matrices belonging to the manifold of symmetric definite positive (SPD) matrices. This interest can also be found in the field of robotics and automatic control, especially to estimate a robot attitude on $SO(3)$ [6], or directly its pose on $SE(3)$, for the problem of simultaneous localization and mapping [7][8]. Also, in blind source separation, the demixing consists in estimating an orthonormal matrix lying on the Stiefel manifold [9].

In order to assess the performance of an estimation problem on a manifold, it is crucial to design an intrinsic error bound which takes into account its properties. To this end, it is necessary to define an error metric preserving the properties of the manifold under consideration. As well as in the Euclidean case, the intrinsic error depends on the estimation strategy and it is fundamental to assess the ultimate achievable estimator performance, information which is brought by the corresponding lower performance bounds. Recently, different intrinsic Cramér-Rao bounds (ICRBs) allowing to lower bound an intrinsic mean square error (IMSE) on a manifold have been proposed in the literature [10][11]. Particularly, closed-form expressions of these theoretical inequalities for the manifold of the SPD matrices have been developed in [12].

In the context of Lie groups (LGs), the ICRB has also been treated in several works. For instance, in [13][14], an inequality for the IMSE on LGs is proposed, and a tractable approximated ICRB, only valid for the $SO(3)$ LG, is provided. In [15][16], a more generic ICRB is also established but only valid for symmetric or isotropic probability density functions (pdfs). Notice that all these bounds have in common to assume that observations lie in an Euclidean space. This is rather restrictive since

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estimation methods with LG observations exist in the literature [17][18]. Consequently, their developments use the Cauchy-Schwartz inequality and integral derivations similar to the Euclidean case proof [19].

Nevertheless, it is known that the classical Euclidean CRB can be derived from a different reasoning. Indeed, the CRB belongs to a class of more generic bounds, where the general Barankin bound (BB) [20][21][22] corresponds to the minimum MSE under uniform unbiasedness constraints [23, Sec. 2]. This bound is in general intractable but, under some unbiasedness restricted assumptions, it can be linked to the CRB as follows: 1) First, the BB can be approximated by the McAulay-Seidman bound (MSB). In this case, the infinite constraint is substituted by a finite linear matrix condition [24] by leveraging on a set of test points lying on the parameters' space, and 2) if restricted to two test points including the unknown value, and tending the limits of the difference between the two test points towards zero, the MSB yields to the classical CRB form [25].

In this work, the main contribution is to adapt the (Euclidean) Barankin's framework for both parameters and observations lying on LGs. By defining the mean square error and bias on LGs in an intrinsic way, we can design a generalization of the Barankin bound on LGs, so-called intrinsic BB (IBB). From the latter, we develop a new intrinsic McAulay-Seidman bound (IMSB), and a new formalization of the ICRB which provides an exact analytical formula valid for any pdf with observations on LGs, contrary to the ICRBs proposed in [13][14]. Both IMSB and ICRB admit closed-form expressions established for the $SO(3)$ and $SE(3)$ LGs, which are derived in this contribution for Euclidean observations. Notice that the latter generalizes the results in [14] [15] [16]. In addition, we take advantage of this formalism to obtain a tractable expression of the ICRB for Gaussian models with observation on LGs, and develop an integral expression of the IBB for the LG $SE(3)$.

The paper is organized as follows: the necessary background on LGs is reminded in Sec. 2. Sec. 3 formalizes the IBB and IMSB, as well as the link with the ICRB, and an integral form of the IBB in the case of $SE(3)$ is obtained. In Sec. 3.4, the closed-form expressions of both IMSB and ICRB for Euclidean and LG observation models, with unknown parameters belonging to the $SE(3)$ LG, are derived. Finally, Sec. 5 validates the proposed IMSB and ICRB by numerical simulations.

2 Background on Lie groups

2.1 Definition

A matrix LG $G \subset \mathbb{R}^{n \times n}$ is a matrix space equipped with a structure of smooth manifold and group.

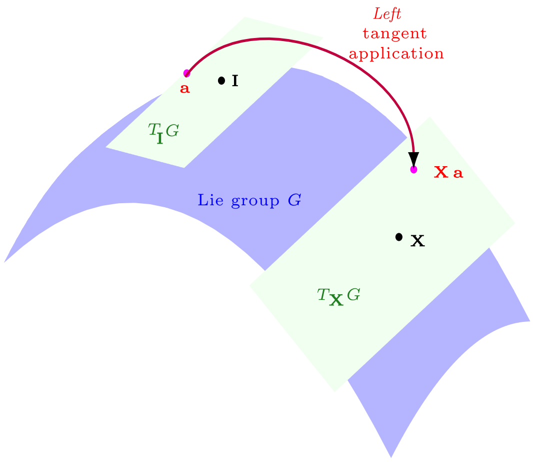

Fig. 1. Relation between the space tangent to $\mathbf{X} \in G$ and the space tangent to the neutral element $\mathbf{I}$. The element $\mathbf{a}$ belonging to $\mathfrak{g} = T_{\mathbf{I}}G$ is transported to $T_{\mathbf{X}}G$ thanks to the left application defined by $\mathbf{Xa}$.

- Its structure of smooth manifold means that it is possible to define the operations of integration and derivation. Particularly, we can specify the notion of tangent space according to each element of G.
- Its structure of group involves the definition of an internal law acting between each element of G. It implies that it exists a neutral element (identity matrix) allowing the inversion of each element. On the other hand, its internal law allows to link each element of the neutral element tangent space to the tangent space of any element, as illustrated in Fig. 1.

2.2 Lie algebra

The tangent space $T_{\mathbf{I}}G$ is called Lie algebra and denoted $\mathfrak{g}$. Its dimension as a space vector defines the intrinsic dimension of the LG. Each element of the LG close enough to the neutral element can be associated to an element of the Lie algebra through the logarithm and exponential applications defined, respectively, by $\text{Exp}_G : \mathfrak{g} \to G$ and $\text{Log}_G : G \to \mathfrak{g}$, as illustrated in Fig. 2. It is known that $\mathfrak{g}$ is isomorph to $\mathbb{R}^m$. Thus, we can define two bijections $[.]^\wedge : \mathbb{R}^m \to \mathfrak{g}$ and $[.]^\vee : \mathfrak{g} \to \mathbb{R}^m$. In this way, we can denote the exponential and logarithm applications such as: $\forall\ \mathbf{a} \in \mathbb{R}^m$, $\text{Exp}_G^\wedge(\mathbf{a}) \ = \text{Exp}_G([\mathbf{a}]_G^\wedge)$ and $\forall\ \mathbf{X} \in G$, $[\text{Log}_G(\mathbf{X})]_G^\vee = \text{Log}_G^\vee(\mathbf{X})$.

2.3 Important LG examples

The two main LGs that will be considered in this work as examples, after the theoretical contribution, are $SO(3)$ and $SE(3)$.

2.3.1 Lie group $SO(3)$

- $SO(3)$ is a group of the set of rotation matrices in 3D space. $\mathbf{R} \in SO(3)$ if and only if $\mathbf{R}\mathbf{R}^\top = \mathbf{I}$ and

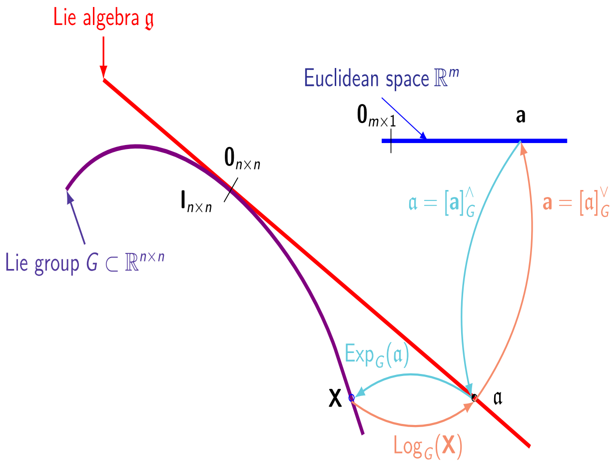

Fig. 2. Relation between $\mathbb{R}^m$, G and $\mathfrak{g}$

$|\mathbf{R}| = 1$.

- Its Lie algebra correspond to the set of skew-symmetric matrices. More precisely, $\mathfrak{so}(3) = \{[\mathbf{w}]_\times | \mathbf{w} \in \mathbb{R}^3\}$ where $[.]_\times$ denotes the operator which transforms a vector to a skew-symmetric matrix.
- Consider $\mathbf{R} \in SO(3)$ such as $\mathbf{R} = \mathrm{Exp}^{\wedge}_{SO(3)}(\mathbf{w})$. The expression of its exponential operator $\mathrm{Exp}^{\wedge}_{SO(3)}(.)$ is given by the Rodrigues formula:

$$\mathrm{Exp}^{\wedge}_{SO(3)}(\mathbf{w}) = \mathbf{I}_{3\times 3} + \frac{[\mathbf{w}]_\times}{\|\mathbf{w}\|}\sin(\|\mathbf{w}\|) + \frac{[\mathbf{w}]^2_\times}{\|\mathbf{w}\|^2}(1-\cos(\|\mathbf{w}\|)). \quad (1)$$

As $\mathrm{Log}^{\vee}_{SO(3)}(\mathbf{R}) = [\mathbf{w}]_\times$, $\mathbf{R} - \mathbf{R}^\top$ provides [26]:

$$\|\mathbf{w}\| = \arccos\left(\frac{\mathrm{tr}(\mathbf{R}) - 1}{2}\right) \quad (2)$$

$$\mathrm{Log}^{\vee}_{SO(3)}(\mathbf{R}) = \frac{\|\mathbf{w}\|\left[\mathbf{R} - \mathbf{R}^\top\right]^\vee}{2\sin(\|\mathbf{w}\|)}. \quad (3)$$

2.3.2 Lie group $SE(3)$

- $SE(3)$ defines the semi-direct product group between $SO(3)$ and $\mathbb{R}^3$. From an application point of view, it can be used to model the pose of a camera or a robot,

$$SE(3) = \left\{\mathbf{M} = \begin{bmatrix}\mathbf{R} & \mathbf{p}\\ \mathbf{0} & 1\end{bmatrix} | \mathbf{R} \in SO(3), \mathbf{p} \in \mathbb{R}^3\right\}. \quad (4)$$

- Its Lie algebra has the following structure:

$$\mathfrak{se}(3) = \left\{\mathbf{A} = \begin{bmatrix}[\mathbf{w}] & \mathbf{u}\\ \mathbf{0} & 0\end{bmatrix} | \mathbf{w} \in \mathbb{R}^3, \mathbf{u} \in \mathbb{R}^3\right\}. \quad (5)$$

- The exponential and logarithm operators can be built from $\mathrm{Log}^{\vee}_{SO(3)}(.)$ and $\mathrm{Exp}^{\wedge}_{SO(3)}(.)$. Indeed, if $\mathbf{M} \in SE(3)$:

$$\mathbf{M} = \begin{bmatrix}\mathbf{R} & \mathbf{x}\\ \mathbf{0} & 1\end{bmatrix} = \mathrm{Exp}^{\wedge}_{SE(3)}(\mathbf{w}), \ \mathbf{w} \in \mathbb{R}^6 \quad (6)$$

with

$$\mathbf{w} = \left[\mathbf{w}_R^\top, \mathbf{w}_p^\top\right], \quad (7)$$

$\mathbf{w}_R$ and $\mathbf{w}_p \in \mathbb{R}^3$, then:

$$\mathrm{Log}^{\vee}_{SE(3)}(\mathbf{M}) = \begin{bmatrix}\mathbf{D}(\mathrm{Log}^{\vee}_{SO(3)}(\mathbf{R}))^{-1}\,\mathbf{p}\\ \mathrm{Log}^{\vee}_{SO(3)}(\mathbf{R})\end{bmatrix}, \quad (8)$$

and for the exponential mapping:

$$\mathrm{Exp}^{\wedge}_{SE(3)}(\mathbf{w}) = \begin{bmatrix}\mathrm{Exp}^{\wedge}_{SO(3)}(\mathbf{w}_R) & \mathbf{D}(\mathrm{Log}^{\vee}_{SO(3)}(\mathbf{R}))\,\mathbf{w}_p\\ 0 & 1\end{bmatrix}, \quad (9)$$

$$\text{with } \mathbf{D}(.) = \mathbf{I}_3 + \frac{1-\cos(\|.\|)}{\|.\|^2}[.]_\times + \frac{\|.\| - \sin(\|.\|)}{\|.\|^3}[.]^2_\times. \quad (10)$$

In the sequel of this section, G denotes a matrix LG and m is its intrinsic dimension.

2.4 Integration and derivation on Lie groups

2.4.1 Integration of a vector-valued function

As a LG is an orientable manifold, it is possible to define a volume form so as to perform integration of a function defined on a matrix LG [27]. Let $f : G \to \mathbb{R}^m$. Thereby, the integral of f on G is defined by

$$I = \int_G f(\mathbf{X})\lambda_G(d\mathbf{X}), \quad (11)$$

where λ_G denotes a volume form called Haar measure. It is a group measure respecting left-invariance properties. Furthermore, as explained in [28], the integral I can be reparametrized as the Riemannian integral on a neighborhood of $\mathbb{R}^m$ as follows:

$$I = c\int_{\mathbb{R}^m} f(\mathrm{Exp}^{\wedge}_G(\boldsymbol{\epsilon}))|\boldsymbol{\phi}_G(-\boldsymbol{\epsilon})|\mathrm{d}\boldsymbol{\epsilon}. \quad (12)$$

where $\boldsymbol{\phi}_G$ corresponds to the left Jacobian matrix of G [29] and c is a real constant. d(.) defines the Lebesgue measure on $\mathbb{R}^6$. In the case of the LG $SE(3)$, this Lebesgue integral is equivalent to a Riemman integral on $\mathbb{R}^6$ [29, Eq. 12.11]. This property will be useful in the following.

2.4.2 Derivation of LG and vector-valued functions

The notion of classical directional derivative can be generalized on LGs. Let κ a LG-valued function on the LG G'. The right Lie derivative [1] of κ in $\mathbf{X} \in G$ is:

$$\mathcal{L}^{R}_{\kappa(\mathbf{X})} = \left. \frac{\partial \text{Log}^{\vee}_{G'} \left(\kappa(\mathbf{X})^{-1} \kappa(\mathbf{X} \, \text{Exp}^{\wedge}_{G} (\boldsymbol{\delta}) \,) \right)}{\partial \boldsymbol{\delta}} \right|_{\boldsymbol{\delta}=\mathbf{0}} \quad \forall \boldsymbol{\delta} \in \mathbb{R}^m. \tag{13}$$

In the case where κ has values in $\mathbb{R}$ or $\mathbb{R}^p$ then $\text{Log}^{\vee}_{G'}(.) = \mathbf{I}$ and the derivative of κ can be defined as

$$\mathcal{L}^{R}_{\kappa(\mathbf{X})} = \left. \frac{\partial \kappa(\mathbf{X} \, \text{Exp}^{\wedge}_{G} (\boldsymbol{\delta}) \,)}{\partial \boldsymbol{\delta}} \right|_{\boldsymbol{\delta}=\mathbf{0}} \quad \forall \boldsymbol{\delta} \in \mathbb{R}^m. \tag{14}$$

2.5 Estimation on Lie groups

Classically, an Euclidean estimator $\widehat{\mathbf{x}}$ of the unknown parameter $\mathbf{x} \in \mathbb{R}^p$, built from the likelihood $p(\mathbf{z}|\mathbf{x})$, can be characterized by three relevant statistical indicators [2]: its intrinsic mean $\mathbf{m}_{\widehat{\mathbf{x}}}$ such as $\int_{\mathbf{z} \in \mathbb{R}^m} (\widehat{\mathbf{x}} - \mathbf{m}_{\widehat{\mathbf{x}}}) p(\mathbf{z}|\mathbf{x}) d\mathbf{z} = 0$, its bias $\int_{\mathbf{z} \in \mathbb{R}^m} (\mathbf{x} - \widehat{\mathbf{x}}) p(\mathbf{z}|\mathbf{x}) \, d\mathbf{z}$ and its mean square error (MSE) $\int_{\mathbf{z} \in \mathbb{R}^m} (\mathbf{x} - \widehat{\mathbf{x}}) \, (\mathbf{x} - \widehat{\mathbf{x}})^{\top} \, p(\mathbf{z}|\mathbf{x}) d\,\mathbf{z}$. Now, consider a random observation $\mathbf{Z}$, belonging to a LG G', depending of an unknown parameter $\mathbf{X} \in G$, and generated by the likelihood $p(\mathbf{Z}|\mathbf{X})$. An estimator of $\mathbf{X}$ is a function $\widehat{\mathbf{X}}(\mathbf{Z}) : G' \to G$.
Intrinsically, the gap between $\mathbf{X}$ and $\widehat{\mathbf{X}}$ can be assessed by the error term $\|\text{Log}^{\vee}_{G} \left(\mathbf{X}^{-1} \widehat{\mathbf{X}} \right) \|$. It should be stressed out that this term defines a geodesic distance only for some LGs, as any semi-simple LGs. Nevertheless, for the LGs $SE(2)$ or $SE(3)$, which are LGs of interest in many applications, it is not built from a LG metric. Even so, it specifies a good indicator of the intrinsic path traveled from $\mathbf{X}$ to $\widehat{\mathbf{X}}$, and it is classically used in the LG estimation literature [14][30].

[1] It is also possible to define a left Lie derivative by switching $\mathbf{X}$ and $\text{Exp}^{\wedge}_{G}(\boldsymbol{\delta})$.
[2] Note that other indicators could be used (for instance the median or the consistency).

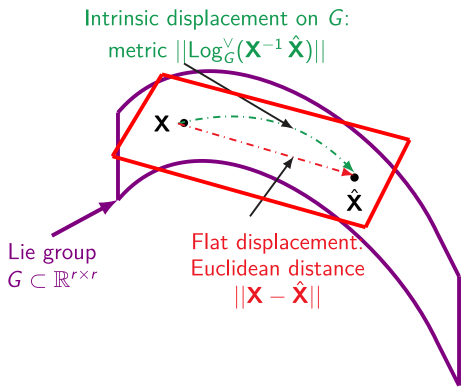

Fig. 3. Illustration of the intrinsic gap between $\mathbf{X}$ and $\widehat{\mathbf{X}}$, which takes into account the curvature of the group.

For the sake of simplicity, we introduce the notation $\boldsymbol{l}_G(\mathbf{X}, \mathbf{Y}) = \text{Log}^{\vee}_{G} \left(\mathbf{X}^{-1} \, \mathbf{Y} \right) \ \forall \mathbf{X}, \mathbf{Y} \in G \times G$. Three intrinsic indicators can be extracted from the estimator $\widehat{\mathbf{X}} \triangleq \widehat{\mathbf{X}}(\mathbf{Z})$:

- Its mean $\mathbf{M}_{\widehat{\mathbf{X}}} \in G$ such that:

$$\int_{G'} \boldsymbol{l}_G \left(\widehat{\mathbf{X}}, \mathbf{M}_{\widehat{\mathbf{X}}} \right) p(\mathbf{Z}|\mathbf{X}) \lambda_G(d\mathbf{Z}) = \mathbf{0} \tag{15}$$

- its intrinsic bias $\mathbf{b}_{\mathbf{Z}|\mathbf{X}} \in \mathbb{R}^m$ given by [14]:

$$\mathbf{b}_{\mathbf{Z}|\mathbf{X}}(\mathbf{X}, \widehat{\mathbf{X}}) = \int_{G'} \boldsymbol{l}_G \left(\mathbf{X}, \widehat{\mathbf{X}} \right) p(\mathbf{Z}|\mathbf{X}) \lambda_G(d\mathbf{Z}) \tag{16}$$

$$\triangleq \mathbb{E}_{p(\mathbf{Z}|\mathbf{X})} \left(\boldsymbol{l}_G \left(\mathbf{X}, \widehat{\mathbf{X}} \right) \right) \tag{17}$$

- its intrinsic MSE (IMSE) $\mathbf{C}_{\mathbf{Z}|\mathbf{X}} \in \mathbb{R}^{m \times m}$ defined by [15]:

$$\mathbf{C}_{\mathbf{Z}|\mathbf{X}}(\mathbf{X}, \widehat{\mathbf{X}}) =$$

$$\int_{G'} \boldsymbol{l}_G \left(\mathbf{X}, \widehat{\mathbf{X}} \right) \boldsymbol{l}_G \left(\mathbf{X}, \widehat{\mathbf{X}} \right)^{\top} p(\mathbf{Z}|\mathbf{X}) \, \lambda_G(d\,\mathbf{Z}) \tag{18}$$

$$\triangleq \mathbb{E}_{p(\mathbf{Z}|\mathbf{X})} \left(\boldsymbol{l}_G \left(\mathbf{X}, \widehat{\mathbf{X}} \right) \boldsymbol{l}_G \left(\mathbf{X}, \widehat{\mathbf{X}} \right)^{\top} \right) \tag{19}$$

3 Expressions of the intrinsic Barankin, McAulay-Seidman and Cramér-Rao bounds

In this section, we introduce novel contributions by developing and formalizing the BB on LGs (i.e., IBB), and its approximation so-called IMSB on LGs. Then, from the latter, we obtain a new expression of the ICRB. As the BB is classically formalized thanks to a uniform unbiasedness condition, we define the IBB by leveraging on the bias given in (17). Afterwards, we restrict the unbiasedness constraint to a set of test points intrinsically on LG, involving the exponential application, what yields to the desired IMSB and ICRB.

3.1 *Background on the Euclidean Barankin, McAulay-Seidman and Cramér-Rao bounds*

Assume a set of observations $\mathbf{z} = \{\mathbf{z}_1, \dots, \mathbf{z}_N\} \in \left(\mathbb{R}^d\right)^N$ depending of $\mathbf{x}_0 \in \mathbb{R}^p$, an unknown parameter vector, and characterized by $p(\mathbf{z}|\mathbf{x}_0)$. Let $\mathbf{g} : \mathbb{R}^p \rightarrow \mathbb{R}^s$ be a smooth function. The BB on the estimator $\widehat{\mathbf{g}(\mathbf{x}_0)}$ is given by [3]:

$$\mathbf{P}_{BB} = \min_{\widehat{\mathbf{g}(\mathbf{x}_0)}} \mathbb{E}_{p(\mathbf{z}|\mathbf{x}_0)} \left(\left(\widehat{\mathbf{g}(\mathbf{x}_0)} - \mathbf{g}(\mathbf{x}_0) \right) \left(\widehat{\mathbf{g}(\mathbf{x}_0)} - \mathbf{g}(\mathbf{x}_0) \right)^\top \right)$$
$$\text{w.r.t. } \mathbb{E}_{p(\mathbf{z}|\mathbf{x})} \left(\widehat{\mathbf{g}(\mathbf{x}_0)} - \mathbf{g}(\mathbf{x}_0) \right) = \mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}_0) \quad \forall \mathbf{x} \in \mathbb{R}^p \tag{20}$$

The above uniform unbiasedness constraint can be seen as a continuum of constraints, consequently, solving the minimization problem (20) is hard. To overcome this issue, $\mathbf{P}_{BB}$ is generally approached by using a set of test points, $\mathbf{x}^{(1:L)} = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(L)}\}$, verifying the unbiasedness condition in (20). Then, the latter can be written as:

$$\mathbb{E}_{p(\mathbf{z}|\mathbf{x}_0)} \left(\mathbf{v}_{\mathbf{x}_0}(\mathbf{x}^{(1:L)}) \left(\mathbf{g}(\mathbf{x}_0) - \widehat{\mathbf{g}\,(\mathbf{x}_0)} \right)^\top \right) \tag{21}$$
$$= \begin{bmatrix} \left(\mathbf{g}(\mathbf{x}^{(1)}) - \mathbf{g}(\mathbf{x}_0)\right)^\top \\ \vdots \\ \left(\mathbf{g}(\mathbf{x}^{(L)}) - \mathbf{g}(\mathbf{x}_0)\right)^\top \end{bmatrix} \tag{22}$$

with $\mathbf{v}_{\mathbf{x}_0}(\mathbf{x}^{(1:L)}) = \left[\frac{p(\mathbf{z}|\mathbf{x}^{(1)})}{p(\mathbf{z}|\mathbf{x}_0)}, \dots, \frac{p(\mathbf{z}|\mathbf{x}^{(L)})}{p(\mathbf{z}|\mathbf{x}_0)} \right]^\top$. This new matrix condition allows to obtain [31, Lemma 1]

$$\mathbb{E}_{p(\mathbf{z}|\mathbf{x}_0)} \left(\left(\widehat{\mathbf{g}(\mathbf{x}_0)} - \mathbf{g}(\mathbf{x}_0) \right) \left(\widehat{\mathbf{g}(\mathbf{x}_0)} - \mathbf{g}(\mathbf{x}_0) \right)^\top \right)$$
$$\succeq \mathbf{\Delta} \mathbf{R}_{\mathbf{v}_{\mathbf{x}_0}}^{-1} \mathbf{\Delta}^\top, \tag{23}$$

where $\succeq$ is defined such that $\forall \mathbf{A}, \mathbf{B},\ \mathbf{A} \succeq \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is a positive definite matrix.
The right-hand term of the inequality is the MSB,

$$\begin{cases} \mathbf{\Delta} = \left[\mathbf{g}(\mathbf{x}^{(1)}) - \mathbf{g}(\mathbf{x}_0), \dots, \mathbf{g}(\mathbf{x}^{(L)}) - \mathbf{g}(\mathbf{x}_0) \right], & (24) \\ \mathbf{R}_{\mathbf{v}_{\mathbf{x}_0}} = \mathbb{E}_{p(\mathbf{z}|\mathbf{x}_0)} \left(\mathbf{v}_{\mathbf{x}_0}(\mathbf{x}^{(1:L)})\, \mathbf{v}_{\mathbf{x}_0}(\mathbf{x}^{(1:L)})^\top \right). & (25) \end{cases}$$

If the test points are written in the following form:

$$\mathbf{x}^{(1)} = \mathbf{x}_0 \tag{26}$$
$$\mathbf{x}^{(l)} = \mathbf{x}_0 + \mathbf{i}_l \delta_l \ \forall l \in \{1, \dots, L-1\} \tag{27}$$

with

$$\mathbf{i}_l = \Big[0, \dots, \underbrace{1}_{l^{\text{th}}\text{component}} \dots, 0 \Big]^\top \in \mathbb{R}^p, \tag{28}$$

then one obtains the CRB expression when $\delta_l \rightarrow 0\, \forall l \in \{1, \dots, L-1\}$ in the right-hand term of (23).

3.2 *Development of the IBB and IMSB on LGs*

Consider now a new set of observations $\mathbf{Z}$ belonging to some matrix Lie group G'. Let be G (with dimension m) and G'' two others LGs. $\mathbf{Z}$ is connected to an unknown parameter $\mathbf{X}_0 \in G$ through its likelihood $p(\mathbf{Z}|\mathbf{X}_0)$. Consider also a LG-valued function $\mathbf{H} : G \rightarrow G''$, and we seek an estimator of $\mathbf{H}(\mathbf{X}_0)$. By using the intrinsic bias definition (17), an intrinsic uniform unbiasedness condition on the LG estimator $\widehat{\mathbf{H}(\mathbf{X}_0)}$ can be generalized,

$$\mathbf{b}_{\mathbf{Z}|\mathbf{X}} \left(\mathbf{H}(\mathbf{X}_0), \widehat{\mathbf{H}(\mathbf{X}_0)} \right) = \boldsymbol{l}_{G''} \left(\mathbf{H}(\mathbf{X}_0), \mathbf{H}(\mathbf{X}) \right) \quad \forall\, \mathbf{X} \in G. \tag{29}$$

Definition 3.2.1 (IBB on LG) *The IBB can be defined as the minimum value of the intrinsic MSE under the intrinsic uniform unbiasedness constraint* (29)*,*

$$\mathbf{P}_{IBB} = \underset{\widehat{\mathbf{H}(\mathbf{X}_0)}}{min}\ \mathbf{C}_{\mathbf{Z}|\mathbf{X}_0}(\mathbf{H}(\mathbf{X}_0), \widehat{\mathbf{H}(\mathbf{X}_0)})$$
$$s.t.\ \mathbf{b}_{\mathbf{Z}|\mathbf{X}} \left(\mathbf{H}(\mathbf{X}_0), \widehat{\mathbf{H}(\mathbf{X}_0)} \right) = \boldsymbol{l}_{G''} \left(\mathbf{H}(\mathbf{X}_0), \mathbf{H}(\mathbf{X}) \right) \quad \forall\, \mathbf{X} \in G \tag{30}$$

Theorem 3.2.1 (IMSB on LG) *Let us consider a set of test points* $\mathbf{X}^{(1:L)} = \{\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(l)}\}$ *verifying* (29)*. Similar to the Euclidean MSB* (23)*, the intrinsic MSE is lower bounded by the so-called IMSB:*

$$\mathbf{C}_{\mathbf{Z}|\mathbf{X}_0}(\mathbf{H}(\mathbf{X}_0), \widehat{\mathbf{H}(\mathbf{X}_0)}) \succeq \mathbf{P}_{IMSB} \tag{31}$$

[3] If we consider two matrices $\mathbf{A}, \mathbf{B}$, $\mathbf{A} \geq \mathbf{B}$ means that the matrix $\mathbf{A} - \mathbf{B}$ is definite positive and $\text{tr}(\mathbf{A} - \mathbf{B}) \geq 0$. Consequently, find the minimum of a matrix $\mathbf{A}$ amounts to find a matrix $\mathbf{A}_{min}$ such for any matrix $\mathbf{B} \neq \mathbf{A}$, $\text{tr}(\mathbf{A} - \mathbf{B}) \geq \text{tr}(\mathbf{A} - \mathbf{A}_{min}) \geq 0$.

$$\mathbf{P}_{IMSB} = \boldsymbol{\Delta}_G \, \mathbf{R}_{\mathbf{v}_{\mathbf{X}_0}}^{-1} \boldsymbol{\Delta}_G^\top \tag{32}$$

$$\boldsymbol{\Delta}_G = \Big[\boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0), \mathbf{H}(\mathbf{X}^{(1)})\right), \dots, \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0), \mathbf{H}(\mathbf{X}^{(L)})\right)\Big] \tag{33}$$

$$\mathbf{R}_{\mathbf{v}_{\mathbf{X}_0}} = \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z}; \mathbf{X}^{(1:L)}\right), \mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z}; \mathbf{X}^{(1:L)}\right)^\top\right), \tag{34}$$

Proof:

The condition (28) applied to the test point provides the following matrix constraint:

$$\mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z}; \mathbf{X}^{(1:L)}\right) \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0), \widehat{\mathbf{H}(\mathbf{X}_0)}\right)^\top\right) = \begin{bmatrix} \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0), \mathbf{H}(\mathbf{X}^{(1)})\right)^\top \\ \vdots \\ \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0), \mathbf{H}(\mathbf{X}^{(L)})\right)^\top \end{bmatrix} \tag{35}$$

with $\mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z}; \mathbf{X}^{(1:L)}\right) = \left[v_{\mathbf{X}_0}\left(\mathbf{Z}; \mathbf{X}^{(1)}\right), \dots, v_{\mathbf{X}_0}\left(\mathbf{Z}; \mathbf{X}^{(L)}\right)\right]^\top$

and $v_{\mathbf{X}_0}\left(\mathbf{Z}; \mathbf{X}^{(l)}\right) = \dfrac{p(\mathbf{Z}|\mathbf{X}^{(l)})}{p(\mathbf{Z}|\mathbf{X}_0)}$.
The quantities involved in the equation (34) are Euclidean and the operator $\mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}$ defines an Euclidean scalar product. Consequently, the Lemma 1 of [31] can be applied to find the formula (30).

3.3 Intrinsic Cramér-Rao bound

In this subsection, we derive the expression of the ICRB. We propose to gather it by generalizing the Euclidean formalism linking the MSB and CRB. It is obtained by defining a set of test points which allows to get a new constraint and a new formulation of the IMSB. Then, the proposed ICRB is derived by letting the test points increment tending towards zero.

Theorem 3.3.1 (ICRB on LG) *Let us consider the following set of test points:*

$$\mathbf{X}^{(1:L)} = \left\{\mathbf{X}_0, \mathbf{X}_0 \operatorname{Exp}_G^\wedge\left(\mathbf{i}_1 \, \delta_1\right), \dots, \mathbf{X}_0 \operatorname{Exp}_G^\wedge\left(\mathbf{i}_{L-1} \, \delta_{L-1}\right)\right\} \tag{36}$$

where $\mathbf{i}_l \, \forall l \in \{1, \dots, L-1\}$ is defined in (28). Consequently,

$$\mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z}; \mathbf{X}^{(1:L)}, \boldsymbol{\delta}\right) = \left[1, \frac{p(\mathbf{Z}|\mathbf{X}_0 \operatorname{Exp}_G^\wedge\left(\mathbf{i}_1 \, \delta_1\right))}{p(\mathbf{Z}|\mathbf{X}_0)}, \dots, \frac{p(\mathbf{Z}|\mathbf{X}_0 \operatorname{Exp}_G^\wedge\left(\mathbf{i}_{L-1} \, \delta_{L-1}\right))}{p(\mathbf{Z}|\mathbf{X}_0)}\right]^\top \text{ with } \boldsymbol{\delta} = [\delta_1, \dots, \delta_{L-1}]^\top \tag{37}$$

$$\boldsymbol{\Delta}_G = \left[\mathbf{0}, \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0), \mathbf{H}(\mathbf{X}_0 \operatorname{Exp}_G^\wedge\left(\mathbf{i}_1 \, \delta_1\right))\right), \dots, \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0), \mathbf{H}(\mathbf{X}_0 \operatorname{Exp}_G^\wedge\left(\mathbf{i}_{L-1} \, \delta_{L-1}\right))\right)\right]. \tag{38}$$

When $\delta_l \to 0$, we obtain the following expression of the ICRB:

$$\mathbf{P}_{ICRB} = \mathcal{L}_{\mathbf{H}(\mathbf{X}_0)}^R \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\mathbf{s}(\mathbf{Z}, \mathbf{X}_0) \, \mathbf{s}(\mathbf{Z}, \mathbf{X}_0)^\top\right)^{-1} \left(\mathcal{L}_{\mathbf{H}(\mathbf{X}_0)}^R\right)^\top. \tag{39}$$

with

$$\mathbf{s}(\mathbf{Z}, \mathbf{X}_0) = \left.\frac{\partial \log p(\mathbf{Z}|\mathbf{X}_0 \operatorname{Exp}_G^\wedge\left(\boldsymbol{\delta}\right))}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}}. \tag{40}$$

and

$$\mathcal{L}_{\mathbf{H}(\mathbf{X}_0)}^R = \left[\frac{\partial \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0), \mathbf{H}(\mathbf{X}_0 \operatorname{Exp}_G^\wedge\left(\mathbf{i}_1 \delta_1\right))\right)}{\partial \delta_1}, \dots, \frac{\partial \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0), \mathbf{H}(\mathbf{X}_0 \operatorname{Exp}_G^\wedge\left(\mathbf{i}_{L-1} \, \delta_{L-1}\right))\right)}{\partial \delta_L}\right] \tag{41}$$

Proof:

First, let us define the matrix $\mathbf{T}$ such as:

$$\mathbf{T}^\top = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\frac{1}{\delta_1} & \frac{1}{\delta_1} & 0 & \cdots & 0 \\ -\frac{1}{\delta_2} & 0 & \frac{1}{\delta_2} & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ -\frac{1}{\delta_{L-1}} & 0 & \ddots & 0 & \frac{1}{\delta_{L-1}} \end{bmatrix} \tag{42}$$

By multiplying to the left the condition (30) by the matrix $\mathbf{T}^\top$, we obtain the new matrix constraint:

$$\mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\tilde{\mathbf{v}}_{\mathbf{X}_0}\left(\mathbf{Z};\mathbf{X}^{(1:L)}\right)\, \boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\widehat{\mathbf{H}(\mathbf{X}_0)}\right)^\top\right) = \begin{bmatrix} \frac{\boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\mathbf{H}(\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\mathbf{i}_1\,\delta_1))\right)}{\delta_1} \\ \vdots \\ \frac{\boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\mathbf{H}(\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\mathbf{i}_{L-1}\,\delta_{L-1}))\right)}{\delta_{L-1}} \end{bmatrix} \quad (43)$$

with

$$\tilde{\mathbf{v}}_{\mathbf{X}_0}\left(\mathbf{Z};\mathbf{X}^{(1:L)},\boldsymbol{\delta}\right) = \left[\frac{p(\mathbf{Z}|\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\mathbf{i}_1\,\delta_1)) - p(\mathbf{Z}|\mathbf{X}_0)}{\delta_1\,p(\mathbf{Z}|\mathbf{X}_0)},\right. \ldots, \left.\frac{p(\mathbf{Z}|\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\mathbf{i}_{L-1}\,\delta_{L-1})) - p(\mathbf{Z}|\mathbf{X}_0)}{\delta_{L-1}\,p(\mathbf{Z}|\mathbf{X}_0)}\right]^\top \quad (44)$$

Thus, by again using [29, Lemma 1], we yield to a new expression of the IMSB:

$$\mathbf{P}_{\mathrm{IMSB}} = \tilde{\boldsymbol{\Delta}}_G\,\tilde{\mathbf{R}}_{\tilde{\mathbf{v}}_{\mathbf{X}_0}}\,\tilde{\boldsymbol{\Delta}}_G^\top \quad (45)$$

with

$$\tilde{\boldsymbol{\Delta}}_G = \left[\frac{\boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\mathbf{H}(\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\mathbf{i}_1\,\delta_1))\,,\right)}{\delta_1}\right. \ldots, \left.\frac{\boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\mathbf{H}(\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\mathbf{i}_{L-1}\,\delta_{L-1}))\right)}{\delta_L}\right] \quad (46)$$

$$\tilde{\mathbf{R}}_{\tilde{\mathbf{v}}_{\mathbf{X}_0}} = \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\tilde{\mathbf{v}}_{\mathbf{X}_0}\left(\mathbf{Z};\mathbf{X}^{(1:L)},\boldsymbol{\delta}\right)\,\tilde{\mathbf{v}}_{\mathbf{X}_0}\left(\mathbf{Z};\mathbf{X}^{(1:L)},\boldsymbol{\delta}\right)^\top\right) \quad (47)$$

Similarly to the Euclidean case, the ICRB can be obtained with $\delta_l \mapsto 0\ \forall l \in \{1,\ldots L-1\}$ in the expression of $\tilde{\mathbf{R}}_{\tilde{\mathbf{v}}_{\mathbf{X}_0}}$ and $\tilde{\boldsymbol{\Delta}}_G$ such as:

$$\mathbf{P}_{\mathrm{ICRB}} = \lim_{\boldsymbol{\delta}\mapsto 0} \tilde{\boldsymbol{\Delta}}_G\,\tilde{\mathbf{R}}_{\tilde{\mathbf{v}}_{\mathbf{X}_0}}^{-1}\,\tilde{\boldsymbol{\Delta}}_G^\top \quad (48)$$

$\tilde{\boldsymbol{\Delta}}_G$ and $\tilde{\mathbf{R}}_{\tilde{\mathbf{v}}_{\mathbf{X}_0}}$ are detailed in the sequel.

• Computation of $\lim_{\boldsymbol{\delta}\mapsto 0} \tilde{\mathbf{R}}_{\tilde{\mathbf{v}}_{\mathbf{X}_0}}$

By defining $\forall \delta \in \mathbb{R}$, $f_{\mathbf{Z},\mathbf{X}_0}(\delta) = p(\mathbf{Z}|\mathbf{X}_0\,\mathrm{Exp}_{G'}^\wedge(\mathbf{i}\,\delta))$, and by definition of the Euclidean derivative, one obtains

$$\lim_{\delta_l\mapsto 0} \frac{f_{\mathbf{Z},\mathbf{X}_0}(\delta_l) - f_{\mathbf{Z},\mathbf{X}_0}(0)}{\delta_l} = \left.\frac{\partial f_{\mathbf{Z},\mathbf{X}_0}(\delta_l)}{\partial \delta_l}\right|_{\delta_l=0} \quad \forall l \in \{1,\ldots L-1\} \quad (49)$$

and

$$\lim_{\delta_l\mapsto 0} \frac{f_{\mathbf{Z},\mathbf{X}_0}(\delta_l) - f_{\mathbf{Z},\mathbf{X}_0}(0)}{\delta_l\, f_{\mathbf{Z},\mathbf{X}_0}(0)} = \left.\frac{\partial \log f_{\mathbf{Z},\mathbf{X}_0}(\delta_l)}{\partial \delta_l}\right|_{\delta_l=0} \quad \forall l \in \{1,\ldots L-1\} \quad (50)$$

Consequently, by concatenating each derivative according to each δ_l, we find that,

$$\lim_{\boldsymbol{\delta}\mapsto 0} \tilde{\mathbf{v}}_{\mathbf{X}_0}\left(\mathbf{Z};\mathbf{X}^{(1:L)},\boldsymbol{\delta}\right) = \mathbf{s}(\mathbf{Z},\mathbf{X}_0) \quad (51)$$

with

$$\mathbf{s}(\mathbf{Z},\mathbf{X}_0) = \left.\frac{\partial \log p(\mathbf{Z}|\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}))}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}}. \quad (52)$$

By assuming that $\tilde{\mathbf{v}}_{\mathbf{X}_0}\left(\mathbf{Z};\mathbf{X}^{(1:L)}\right)$ is upper-bounded by a continuous integrable function, then

$$\boldsymbol{\delta} \mapsto \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\tilde{\mathbf{v}}_{\mathbf{X}_0}\left(\mathbf{Z};\mathbf{X}^{(1:L)},\boldsymbol{\delta}\right)\,\tilde{\mathbf{v}}_{\mathbf{X}_0}\left(\mathbf{Z};\mathbf{X}^{(1:L)},\boldsymbol{\delta}\right)^\top\right)$$

is continuous and

$$\lim_{\boldsymbol{\delta}\mapsto 0} \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\tilde{\mathbf{v}}_{\mathbf{X}_0}\left(\mathbf{Z};\mathbf{X}^{(1:L)},\boldsymbol{\delta}\right)\,\tilde{\mathbf{v}}_{\mathbf{X}_0}\left(\mathbf{Z};\mathbf{X}^{(1:L)},\boldsymbol{\delta}\right)^\top\right) = \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\mathbf{s}(\mathbf{Z},\mathbf{X}_0)\,\mathbf{s}(\mathbf{Z},\mathbf{X}_0)^\top\right). \quad (53)$$

Finally, we have

$$\lim_{\boldsymbol{\delta}\mapsto 0} \tilde{\mathbf{R}}_{\tilde{\mathbf{v}}_{\mathbf{X}_0}} = \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\mathbf{s}(\mathbf{Z},\mathbf{X}_0)\,\mathbf{s}(\mathbf{Z},\mathbf{X}_0)^\top\right). \quad (54)$$

• Computation of $\lim_{\boldsymbol{\delta}\mapsto 0} \tilde{\boldsymbol{\Delta}}_G$

In a similar fashion as in the previous case, one has

$$\lim_{\delta_i\mapsto 0} \frac{\boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\mathbf{H}\left(\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\mathbf{i}_i\,\delta_i)\right)\right)}{\delta_i} = \left.\frac{\partial\,\boldsymbol{l}_{G''}\left(\mathbf{H}(\mathbf{X}_0),\mathbf{H}\left(\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\mathbf{i}_i\,\delta_i)\right)\right)}{\partial \delta_i}\right|_{\delta_i=0} \quad (55)$$

By definition, the last quantity corresponds to the i^{th} column of the *right Lie derivative* of $\mathbf{H}$ according to $\mathbf{X}_0$. Consequently, it comes

$$\lim_{\boldsymbol{\delta}\mapsto 0} \tilde{\boldsymbol{\Delta}}_G = \mathcal{L}^R_{\mathbf{H}(\mathbf{X}_0)} \quad (56)$$

By substituting (54) and (56) in (48), we obtain the desired formula.

Remark 3.3.1 It should be stressed out that if the considered observation LG G' is unimodular [15] [29] (which is the case for instance for $\mathbb{R}^p$, $SO(3)$ and $SE(3)$), and if the function $\boldsymbol{\delta} \to \log(p(\mathbf{Z}|\mathbf{X}_o \mathrm{Exp}_G^{\wedge}(\boldsymbol{\delta})\ )$ is sufficiently regular [19], then the expression (54) is:

$$\mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\mathbf{s}(\mathbf{Z},\mathbf{X}_0)\,\mathbf{s}(\mathbf{Z},\mathbf{X}_0)\right) = \\ -\mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\left.\frac{\partial^2 \log p(\mathbf{Z}|\mathbf{X}_0\,\mathrm{Exp}_G^{\wedge}(\boldsymbol{\delta}_1)\ \mathrm{Exp}_G^{\wedge}(\boldsymbol{\delta}_2))}{\partial\,\boldsymbol{\delta}_1\,\partial\,\boldsymbol{\delta}_2}\right|_{\boldsymbol{\delta}_1,\boldsymbol{\delta}_2=\mathbf{0}}\right) \tag{57}$$

The advantage of this formula is that it allows to retrieve closed-form expressions of the bound in an easier way.

3.4 Link with the state-of-the-art ICRB

When $\mathbf{H} = \mathbf{I}$ and the set of observations is Euclidean, we remark that expression (39) corresponds to the approached ICRB developed in [14], gathered by neglecting the terms linked to the curvature of the group. Indeed, the bound in [14] has the following form:

$$\begin{aligned}\tilde{\mathbf{P}} =& \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\boldsymbol{\psi}_G\left(\boldsymbol{l}_G(\mathbf{X}_0,\widehat{\mathbf{X}}_0)\right)\right)\mathbf{P}_{ICRB} \\ & \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\boldsymbol{\psi}_G\left(\boldsymbol{l}_G(\mathbf{X}_0,\widehat{\mathbf{X}}_0)\right)\right)^{\top}.\end{aligned} \tag{58}$$

where $\boldsymbol{\psi}_G(.)$ is the inverse of the left Jacobian matrix $\boldsymbol{\phi}_G(.)$, and $\mathbf{P}_{ICRB}$ is defined by equation (39). When $\widehat{\mathbf{X}}_0$ is assumed to be close to $\mathbf{X}_0$, $\boldsymbol{\psi}_G(.)$ can be approximated by the identity matrix, and we retrieve equation (57). In our demonstration, (57) is established without approximations: this proves that $\mathbf{P}_{ICRB}$ is also a lower bound, as well as $\tilde{\mathbf{P}}$, which additionally allows to break free of the considered LG local geometry, because the curvature terms linking to $\boldsymbol{\psi}_G(.)$ are not taken into account. It is worth noticing that the bound $\tilde{\mathbf{P}}$ (58) is not explicit. Indeed, the right-hand term of (58) depends itself of the IMSE through the term $\boldsymbol{l}_G(\mathbf{X}_0,\widehat{\mathbf{X}}_0)$. Consequently, its implementation is not trivial and requires the approximation of the term $\boldsymbol{\psi}_G(.)$.

4 IMSB and ICRB analytical expressions

In the following, we develop analytical expressions of the proposed IMSB and ICRB.

- First, we consider a model that leads to tractable ICRB and IMSB. To succeed that, we assume a Gaussian observation model that can be written from unknown parameters belonging to $SO(3)$ or $SE(3)$. Particularly, we detail the computations leading to the closed-form expression of the IMSB, then those of the ICRB. In the case of $SO(3)$, we observe that the obtained ICRB is a generalization of the ICRB in [32].
- Then, we propose a more generic model on any matrix LGs where only the ICRB is tractable. Observations are on LGs and are assumed non-linear.
- We conclude this section by demonstrating an integral expression of the IBB in the case of $SE(3)$.

For the sake of simplicity and without loss of generality, we consider that the unknown parameter is directly $\mathbf{X}_0$ so that $\mathbf{H} = \mathbf{I}$.

4.1 Expressions for $SO(3)$ and $SE(3)$ with linear Euclidean observations

The proposed model has the following form,

$$\mathbf{z}_n = \mathbf{\Pi}\,\mathbf{X}_0\,\mathbf{\Pi}'\mathbf{p}_n + \mathbf{n}_n \quad \mathbf{n}_n \sim \mathcal{N}_{\mathbb{R}^3}(\mathbf{0},\mathbf{\Sigma}) \quad \forall n \in \{1,\dots N\} \tag{59}$$

where $\{\mathbf{z}_n\}_{n=1}^N$ and $\{\mathbf{p}_n\}_{n=1}^N$ are sets of N 3D points. $\mathbf{X}_0 \in G$ is the unknown parameter modeling a geometrical transformation between them, and belonging to either $SO(3)$ or $SE(3)$. $\{\mathbf{n}_n\}_{n=1}^N$ are assumed to be independent among them. $\mathbf{\Pi}$ and $\mathbf{\Pi}'$ are two affine operators. The full observation vector is $\mathbf{z} = \left[\mathbf{z}_1^\top,\dots,\mathbf{z}_N^\top\right]^\top$.

- In the case of $G = SO(3)$, the model is computed with $\mathbf{\Pi} = \mathbf{\Pi}' = \mathbf{I}_3$. This corresponds to the Wahba's problem [33].
- In the case of $G = SE(3)$, the model is computed with $\mathbf{\Pi} = [\mathbf{I}_3, 0]$ and $\mathbf{\Pi}'$ is defined such as $\mathbf{\Pi}'\,\mathbf{x} = [\mathbf{x}, 1]^\top\ \forall \mathbf{x} \in \mathbb{R}^3$. It can be identified to the well-known Perspective-n-Points problem in computer vision [34]. Classically, the observations correspond to pixel measurements [35], but in order to simply the problem, we do not consider the intrinsic transformation between 3D and pixel measurements.

Theorem 4.1.1 (IMSB for Euclidean observations) *The IMSB for the model (58) is given by the formula*

(32) *with:*

$$\left[\mathbf{R}_{\mathbf{v}_{\mathbf{X}_0}}\right]_{i,j} = \exp\left(0.5\ \left(\mathbf{m}_{ij}^\top\ (\mathbf{I}_N \otimes \boldsymbol{\Sigma})\,\mathbf{m}_{ij} - \delta_{i,j}\right)\right), \tag{60}$$

$$\mathbf{m}_{ij} = (\mathbf{I}_N \otimes \boldsymbol{\Sigma})^{-1}\left(\mathbf{I}_N \otimes \boldsymbol{\Pi}\,\mathbf{X}^{(i)}\,\boldsymbol{\Pi}'\right)\mathbf{p} + (\mathbf{I}_N \otimes \boldsymbol{\Sigma})^{-1}\left(\mathbf{I}_N \otimes \boldsymbol{\Pi}\mathbf{X}^{(j)}\boldsymbol{\Pi}'\right)\mathbf{p} - (\mathbf{I}_N \otimes \boldsymbol{\Sigma})\ (\mathbf{I}_N \otimes \boldsymbol{\Pi}\mathbf{X}_0\boldsymbol{\Pi}')\ \mathbf{p}, \tag{61}$$

$$\delta_{ij} = \left(\left(\mathbf{I}_N \otimes \boldsymbol{\Pi}\,\mathbf{X}^{(i)}\,\boldsymbol{\Pi}'\right)\mathbf{p}\right)^\top (\mathbf{I}_N \otimes \boldsymbol{\Sigma})^{-1}\left(\left(\mathbf{I}_N \otimes \boldsymbol{\Pi}\,\mathbf{X}^{(i)}\,\boldsymbol{\Pi}'\right)\mathbf{p}\right) + \left(\left(\mathbf{I}_N \otimes \boldsymbol{\Pi}\,\mathbf{X}^{(j)}\,\boldsymbol{\Pi}'\right)\mathbf{p}\right)^\top (\mathbf{I}_N \otimes \boldsymbol{\Sigma})^{-1}\left(\left(\mathbf{I}_N \otimes \boldsymbol{\Pi}\mathbf{X}^{(j)}\boldsymbol{\Pi}'\right)\mathbf{p}\right) - ((\mathbf{I}_N \otimes \boldsymbol{\Pi}\,\mathbf{X}_0\,\boldsymbol{\Pi}')\ \mathbf{p})^\top\ (\mathbf{I}_N \otimes \boldsymbol{\Sigma})^{-1}\ ((\mathbf{I}_N \otimes \boldsymbol{\Pi}\,\mathbf{X}_0\,\boldsymbol{\Pi}')\ \mathbf{p})\,. \tag{62}$$

Proof:

From (31), $\mathbf{R}_{\mathbf{v}_{\mathbf{X}_0}}$ is given by:

$$\mathbf{R}_{\mathbf{v}_{\mathbf{X}_0}} = \mathbb{E}_{p(\mathbf{z}|\mathbf{X}_0)}\left(\mathbf{v}_{\mathbf{X}_0}\left(\mathbf{z};\mathbf{X}^{(1:L)}\right)\mathbf{v}_{\mathbf{X}_0}\left(\mathbf{z};\mathbf{X}^{(1:L)}\right)^\top\right),$$

According to (59), and by independence of $\{\mathbf{n}_n\}_{n=1}^N$, one has

$$p(\mathbf{z}_n|\mathbf{X}_0) = \mathcal{N}(\boldsymbol{\Pi}\,\mathbf{X}_0\boldsymbol{\Pi}'\,\mathbf{p}_n, \boldsymbol{\Sigma})\ \forall n \in \{1,\dots,N\},$$
$$p(\mathbf{z}|\mathbf{X}_0) = \mathcal{N}(\mathbf{I}_N \otimes \boldsymbol{\Pi}\,\mathbf{X}_0\boldsymbol{\Pi}'\,\mathbf{p}, \mathbf{I}_N \otimes \boldsymbol{\Sigma}),$$

with $\mathbf{p} = \left[\mathbf{p}_1^\top,\dots,\mathbf{p}_N^\top\right]^\top$ and $\otimes$ the Kronecker product. $\forall(i,j) \in \{1,\dots L\}^2$, we can show by usual manipulations of Gaussian distributions that $\mathbf{R}_{\mathbf{v}_{\mathbf{X}_0}}$ verifies (61) [25, Eq. B.2d]:

Remark 4.1.1 We remind that $\boldsymbol{\Delta}_G$ is written as:

$$\boldsymbol{\Delta}_G = \left[\boldsymbol{l}_G\left(\mathbf{X}_0, \mathbf{X}^{(1)}\right),\dots,\boldsymbol{l}_G\left(\mathbf{X}_0, \mathbf{X}^{(L)}\right)\right]. \tag{63}$$

To compute it, we need to know the expression of $\mathrm{Log}_G^\vee(.)$ in the case of $SO(3)$ and $SE(3)$. They are provided in Sec. 2.3.

Theorem 4.1.2 (ICRB for Euclidean observations) *The ICRB for the model (58) is given by:*

$$\mathbf{P}_{ICRB} = \left(\mathcal{J}^{(1)}\right)^{-1} \tag{64}$$

with:

$$\mathcal{J}_{(s,q)}^{(1)} = \sum_{n=1}^N \mathbf{p}_n^\top \boldsymbol{\Pi}'^\top\,\mathbf{B}_{(s,q)}\,\boldsymbol{\Pi}'\,\mathbf{p}_n \quad \forall(s,q) \in \{1,\dots,m\}^2, \tag{65}$$

and:

$$\mathbf{B}_{(s,q)} = \frac{1}{2}\left(\mathbf{G}_s^\top\mathbf{G}_q^\top\mathbf{A} + \mathbf{G}_s^\top\mathbf{A}\,\mathbf{G}_q + \mathbf{G}_q^\top\mathbf{A}\mathbf{G}_s - \mathbf{A}\,\mathbf{G}_s\mathbf{G}_q\right) \tag{66}$$

$$\mathbf{A} = (\boldsymbol{\Pi}\,\mathbf{X}_0)^\top\boldsymbol{\Sigma}^{-1}\,(\boldsymbol{\Pi}\,\mathbf{X}_0) \tag{67}$$

where $\{\mathbf{G}_s\}_{s=1}^m$ corresponds to a basis of the Lie algebra $\mathfrak{se}(3)$ or $\mathfrak{so}(3)$ and m is the intrinsic dimension of G.

Proof:

The ICRB for $SO(3)$ and $SE(3)$ can be obtained by the formula (57). Thereby, its computation requires two steps:

- to derive the logarithm of $p(\mathbf{z}|\mathbf{X}_0)$,
- to differentiate the gathered expression twice according to $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$.

(1) According to (59), and by independence of $\{\mathbf{n}_n\}_{n=1}^N$, we can write the log-likelihood of $\mathbf{z} = \{\mathbf{z}_n\}_{n=1}^N$ as:

$$\log p(\mathbf{z}|\mathbf{X}_0) = \sum_{n=1}^N \|\mathbf{z}_n - \boldsymbol{\Pi}\,\mathbf{X}_0\boldsymbol{\Pi}'\,\mathbf{p}_n\|_{\boldsymbol{\Sigma}}^2. \tag{68}$$

According to (57), the ICRB can be deduced by computing the following quantity:

$$\left.\frac{\partial^2 \log p(\mathbf{z}|\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)\ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)\ )}{\partial\boldsymbol{\delta}_1\,\partial\boldsymbol{\delta}_2}\right|_{\boldsymbol{\delta}_1,\boldsymbol{\delta}_2=\mathbf{0}} = \left.\frac{\partial^2 \sum_{n=1}^N \|\mathbf{z}_n - \boldsymbol{\Pi}\,\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)\,\boldsymbol{\Pi}'\,\mathbf{p}_n\|_{\boldsymbol{\Sigma}}^2}{\partial\boldsymbol{\delta}_1\,\partial\boldsymbol{\delta}_2}\right|_{\boldsymbol{\delta}_1,\boldsymbol{\delta}_2=\mathbf{0}} \tag{69}$$

which leads to

$$\|\mathbf{z}_n - \boldsymbol{\Pi}\,\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)\ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)\ \boldsymbol{\Pi}'\,\mathbf{p}_n\|_{\boldsymbol{\Sigma}}^2 = \|\mathbf{z}_n\|_{\boldsymbol{\Sigma}}^2 - 2\,\mathbf{z}_n^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\Pi}\,\mathbf{X}_0\ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)\ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)\ \boldsymbol{\Pi}'\,\mathbf{p}_n + \mathbf{p}_n^\top\boldsymbol{\Pi}'^\top\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)^\top\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)^\top\,\mathbf{A} \times \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)\ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)\ \boldsymbol{\Pi}'\,\mathbf{p}_n, \tag{70}$$

with $\mathbf{A} = (\boldsymbol{\Pi}\,\mathbf{X}_0)^\top\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Pi}\,\mathbf{X}_0)$.

(2) Then, by differentiating (70) successively with respect to $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$, and by taking their values to $\mathbf{0}$, we can show (refer to App. B) that the ICRB is equal to (63).

4.1.1 Particular case of SO(3)

In the case of $G = SO(3)$, and $\boldsymbol{\Sigma} = \sigma^2\ \mathbf{I}_3$, we have $\mathbf{A} = \frac{1}{\sigma^2}\mathbf{X}^\top\mathbf{X} = \frac{1}{\sigma^2}\mathbf{I}$. We show that,

$$\mathbf{B}_{(s,q)} = \frac{1}{\sigma^2}\mathbf{G}_s^\top\mathbf{G}_q\ \forall(s,q)\in\{1,\dots,3\}^2. \tag{71}$$

Consequently, the matrix $\mathcal{J}^{(1)}$ reads,

$$\mathcal{J}^{(1)}_{(s,q)} = \frac{1}{\sigma^2}\sum_{n=1}^{N}\mathbf{p}_n^\top\,\mathbf{G}_s^\top\mathbf{G}_q\,\mathbf{p}_n\ \forall(s,q)\in\{1,\dots,3\}^2. \tag{72}$$

As $\{\mathbf{G}_s\}_{s=1}^3$ is a basis of the set of skew-symmetric matrix, then,

$$\mathcal{J}^{(1)} = \frac{1}{\sigma^2}\sum_{n=1}^{N}[\mathbf{p}_n]_\times^\top\ [\mathbf{p}_n]_\times\,. \tag{73}$$

The inverse of the last expression corresponds to the closed-form ICRB on $SO(3)$ in [32, Prop. 2].

4.2 Expression of the ICRB for non-linear observations on LGs

Now, we assume a set of independent observations $\{\mathbf{Z}_1,\dots\mathbf{Z}_N\}$ belonging to some LG G' and following a concentrated Gaussian distribution [36]:

$$\mathbf{Z}_i = \mathbf{F}(\mathbf{X}_0)\,\mathrm{Exp}_{G'}^\wedge(\mathbf{n}_i)\quad \mathbf{n}_i\sim\mathcal{N}(\mathbf{0},\boldsymbol{\Sigma})\ \ \forall i\in\{1,\dots,N\} \tag{74}$$

where $\mathbf{F}: G\to G'$ is a potentially non-linear and smooth function.

Theorem 4.2.1 (ICRB for LG observations) *The ICRB for the model* (74) *is given by:*

$$\mathbf{P}_{ICRB} = \left(\mathcal{J}^{(2)}\right)^{-1} \tag{75}$$

with

$$\mathcal{J}^{(2)} = \sum_{i=1}^{N}\left(\mathcal{L}^R_{\mathbf{F}(\mathbf{X}_0)}\right)^\top\times \mathbb{E}\left(\boldsymbol{\psi}_{G'}(\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0),\mathbf{Z}_i))^\top\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\psi}_{G'}(\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0),\mathbf{Z}_i))\right)\mathcal{L}^R_{\mathbf{F}(\mathbf{X}_0)} \tag{76}$$

Proof:

We remind that the intrinsic Fisher information matrix has the following generic expression:

$$\mathcal{J}^{(2)} = -\mathbb{E}\left(\frac{\partial^2\log p\left(\mathbf{Z}|\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)\ \ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)\,\right)}{\partial\boldsymbol{\delta}_1\partial\boldsymbol{\delta}_2}\right)\Bigg|_{\boldsymbol{\delta}_1,\boldsymbol{\delta}_2=\mathbf{0}} \tag{77}$$

First, we can write that the log-likelihood:

$$\log p\left(\mathbf{Z}|\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)\ \ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)\,\right) = \overbrace{K}^{\in\mathbb{R}} - \frac{1}{2}\sum_{i=1}^{N}\left\|\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)\ \ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)\ ,\mathbf{Z}_i)\right\|^2_{\boldsymbol{\Sigma}}\quad \forall\boldsymbol{\delta}_1,\boldsymbol{\delta}_2\in(\mathbb{R}^p)^2 \tag{78}$$

$\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ are assumed sufficiently small so that:

$$\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)\ \ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)\ = \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1+\boldsymbol{\delta}_2) \tag{79}$$

Thus

$$\log p\left(\mathbf{Z}|\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)\ \ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)\,\right) = \overbrace{K}^{\in\mathbb{R}} - \frac{1}{2}\sum_{i=1}^{N}\left\|\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1+\boldsymbol{\delta}_2)\ ,\mathbf{Z}_i)\right\|^2_{\boldsymbol{\Sigma}}.) \tag{80}$$

Now, let us consider a first order Taylor-Young expansion on LG of the function $\mathbf{X}\to\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}),.)$ [29]

$$\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})\,),.) = \boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0),.) + \frac{\partial\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\epsilon})\ ,.)}{\partial\boldsymbol{\epsilon}}^\top\Bigg|_{\boldsymbol{\epsilon}=\mathbf{0}}\boldsymbol{\delta} + O(\|\boldsymbol{\delta}\|^2) \tag{81}$$

According to the Baker-Campbell-Hausdorff formula [37], we know that:

$$\frac{\partial\,\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\epsilon})\ ,\mathbf{Z}_i))}{\partial\boldsymbol{\epsilon}}\Bigg|_{\boldsymbol{\epsilon}=\mathbf{0}} = \boldsymbol{\psi}_{G'}(l_{G'}(\mathbf{F}(\mathbf{X}_0),\mathbf{Z}_i)) \tag{82}$$

Consequently, by composition of LG derivatives of $\boldsymbol{\epsilon}\to\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\epsilon})\ \ ,)$ and $\boldsymbol{\epsilon}\to\mathbf{F}(\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\epsilon})\ )$, we have:

$$\frac{\partial\,\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\epsilon})\,),\mathbf{Z}_i)}{\partial\boldsymbol{\epsilon}}\Bigg|_{\boldsymbol{\epsilon}=\mathbf{0}} = \boldsymbol{\psi}_{G'}(l_{G'}(\mathbf{F}(\mathbf{X}_0),\mathbf{Z}_i))\,\mathcal{L}^R_{\mathbf{F}(\mathbf{X}_0)} \tag{83}$$

By injecting the latter in (81), one obtains:

$$\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0\,\mathrm{Exp}_G^\wedge(\boldsymbol{\delta})\,) = \boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0)) + \left(\mathcal{L}^R_{\mathbf{F}(\mathbf{X}_0)}\right)^\top\boldsymbol{\psi}_{G'}(l_{G'}(\mathbf{F}(\mathbf{X}_0),\mathbf{Z}_i))^\top\boldsymbol{\delta} + O(\|\boldsymbol{\delta}\|^2) \tag{84}$$

Thus, as $\|\boldsymbol{\delta}_1+\boldsymbol{\delta}_2\| \to 0$, the equation (79) can be written as:

$$\log p\left(\mathbf{Z}|\mathbf{X}_0 \operatorname{Exp}_G^\wedge(\boldsymbol{\delta}_1)\ \operatorname{Exp}_G^\wedge(\boldsymbol{\delta}_2)\right) = \overbrace{K}^{\in\mathbb{R}} - \frac{1}{2}\sum_{i=1}^{N} \left\|\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0 \operatorname{Exp}_G^\wedge(\boldsymbol{\delta}_1+\boldsymbol{\delta}_2)\ , \mathbf{Z}_i)\right\|_{\boldsymbol{\Sigma}}^2$$
$$= \overbrace{K}^{\in\mathbb{R}} - \frac{1}{2}\sum_{i=1}^{N} \left\|\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0)) + \left(\mathcal{L}^R_{\mathbf{F}(\mathbf{X}_0)}\right)^\top \times \boldsymbol{\psi}_{G'}(\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0), \mathbf{Z}_i))^\top(\boldsymbol{\delta}_1+\boldsymbol{\delta}_2) + O(\|\boldsymbol{\delta}_1+\boldsymbol{\delta}_2\|^2)\right\|_{\boldsymbol{\Sigma}}^2 \quad (85)$$

By developing the previous expression, differentiating according to $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ and taking it value to $\mathbf{0}$, we gather:

$$\left.\frac{\partial^2 \log p\left(\mathbf{Z}|\mathbf{X}_0 \operatorname{Exp}_G^\wedge(\boldsymbol{\delta}_1)\ \operatorname{Exp}_G^\wedge(\boldsymbol{\delta}_2)\right)}{\partial\boldsymbol{\delta}_1\partial\boldsymbol{\delta}_2}\right|_{\boldsymbol{\delta}_1=\boldsymbol{\delta}_2=\mathbf{0}} = -\sum_{i=1}^{N}\left(\mathcal{L}^R_{\mathbf{F}(\mathbf{X}_0)}\right)^\top \boldsymbol{\psi}_{G'}(\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0), \mathbf{Z}_i)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\psi}_{G'}(\boldsymbol{l}_{G'}(\mathbf{F}(\mathbf{X}_0), \mathbf{Z}_i))\,\mathcal{L}^R_{\mathbf{F}(\mathbf{X}_0)} \quad (86)$$

By taking the mean value, we obtain the following expression of $\mathcal{J}^{(2)}$ in (76).

Remark 4.2.1 The last expression can be interpreted as a generalization of the Slepian-Bangs formula, which a well-known formula for the Fisher information matrix in the Euclidean case under Gaussian Euclidean observations [38] [39]. Especially, when G and G' are Euclidean space, $\boldsymbol{\psi}_{G'}(.) = \mathbf{I}$, $\operatorname{Log}_{G'}^\vee(.) = \begin{bmatrix}\mathbf{0} & . \\ \mathbf{0} & 0\end{bmatrix}$, $\operatorname{Exp}_G^\wedge(.) = \begin{bmatrix}\mathbf{I} & . \\ \mathbf{0} & 0\end{bmatrix}$, then the model (73) becomes Gaussian Euclidean and we retrieve the Euclidean formula.

4.3 Integral form of the IBB on $SE(3)$

In the Euclidean case, an integral form of the BB has been proposed in [40]. It can be relevant and useful especially when the MSB is not tractable. Indeed, if this integral form could be developed with a closed-form, the BB could be implemented numerically. This form can be obtained by leveraging on the MSB expression. Indeed, by considering linear transformations on the unbiasedness constraint at test-points, one can obtain a lower bound which the limit when the number of transformations and points tends to infinity provides an integral form of the BB (not released in [17]) [40].

In this part, we propose to transpose this result for the LG $SE(3)$. In the Euclidean case, the reasoning is handled by using the property that an Euclidean integral can be approached by a sum on several sub-intervals of the definition set. This result is in general not available for parameters lying on LGs. Nevertheless, thanks to the structure of $SE(3)$, we can deal with Euclidean parameters in order to determine the IBB of this specific group and carry out an equivalent reasoning.

Let us consider a parametrization of each test points under the form $\mathbf{X}^{(l)} = \operatorname{Exp}_{SE(3)}^\wedge(\boldsymbol{\epsilon}_l)$ $(\boldsymbol{\epsilon}_l \in \mathbb{R}^6)$. A linear transformation constraint on the unbiasedness condition (43) can be derived on $SE(3)$:

$$\mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\mathbf{h}^\top \mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z}; \{\operatorname{Exp}_{SE(3)}^\wedge(\boldsymbol{\epsilon}_l)\ \}\right) \boldsymbol{l}_{SE(3)}(\mathbf{X}_0, \widehat{\mathbf{X}}_0)^\top\right) = \mathbf{h}^\top \begin{bmatrix} \boldsymbol{l}_{SE(3)}\left(\mathbf{X}_0, \operatorname{Exp}_{SE(3)}^\wedge(\boldsymbol{\epsilon}_1)\ \right)^\top \\ \vdots \\ \boldsymbol{l}_{SE(3)}\left(\mathbf{X}_0, \operatorname{Exp}_{SE(3)}^\wedge(\boldsymbol{\epsilon}_L)\ \right)^\top \end{bmatrix}, \quad (87)$$

where the transformation $\mathbf{h} \in \mathbb{R}^L$ is defined as:

$$\mathbf{h}_k = \left[h_1|\boldsymbol{\phi}_{SE(3)}(-\boldsymbol{\delta}_1)|, \dots, h_L|\boldsymbol{\phi}_{SE(3)}(-\boldsymbol{\epsilon}_L)|\right]^\top \quad \forall k \in \{1, \dots, K\}. \quad (88)$$

If we define a parametric function h on the set $S \times SE(3)$, where $S = \left\{\boldsymbol{\Upsilon} \in SE(3) | \Upsilon = \begin{bmatrix}\mathbf{I} & \boldsymbol{\tau} \\ \mathbf{0} & 1\end{bmatrix}\right\}$, such as $h_l = h(\boldsymbol{\Upsilon}, \operatorname{Exp}_{SE(3)}^\wedge(\boldsymbol{\epsilon}_l)\ ), \forall l \in \{1, \dots, L-1\}$. If K transformations are considered, then $\mathbf{h}$ can be indexed by k:

$$\mathbf{h}_k = \left[h(\boldsymbol{\Upsilon}_k, \operatorname{Exp}_{SE(3)}^\wedge(\boldsymbol{\epsilon}_l)\ )\,|\boldsymbol{\phi}_{SE(3)}(-\boldsymbol{\delta}_1)|, \dots, h(\boldsymbol{\Upsilon}_k, \operatorname{Exp}_{SE(3)}^\wedge(\boldsymbol{\epsilon}_L)\ )\,|\boldsymbol{\phi}_{SE(3)}(-\boldsymbol{\epsilon}_L)|\right]. \quad (89)$$

4.3.1 Limit of the IMSB when $L \to \infty$

When the number of points $\{\boldsymbol{\epsilon}_l\}_{l=1}^L$ tends to infinity, the sums iterating on ϵ_l in the scalar products of the constraint (87) converge to Riemannian integrals on $\mathbb{R}^6$:

$$\mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(q(\boldsymbol{\Upsilon}, \mathbf{Z})\,\boldsymbol{l}_{SE(3)}\left(\mathbf{X}_0, \widehat{\mathbf{X}}_0\right)^\top\right) = \boldsymbol{\Gamma}(\boldsymbol{\Upsilon}) \quad (90)$$

with

$$\begin{cases} q(\boldsymbol{\Upsilon},\mathbf{Z}) = \int_{\mathbb{R}^6} h(\boldsymbol{\Upsilon}, \mathrm{Exp}^\wedge_{SE(3)}(\boldsymbol{\epsilon}))|\boldsymbol{\phi}_{SE(3)}(-\boldsymbol{\epsilon})| \times \\ v_{\mathbf{X}_0}(\mathbf{Z}; \mathrm{Exp}^\wedge_{SE(3)}(\boldsymbol{\epsilon}))\,\mathrm{d}\boldsymbol{\epsilon} & (91) \\ \boldsymbol{\Gamma}(\boldsymbol{\Upsilon}) = \int_{\mathbb{R}^6} h(\boldsymbol{\Upsilon}, \mathrm{Exp}^\wedge_{SE(3)}(\boldsymbol{\epsilon}))|\boldsymbol{\phi}_{SE(3)}(-\boldsymbol{\epsilon})| \times \\ \mathbf{c}(\mathrm{Exp}^\wedge_{SE(3)}(\boldsymbol{\epsilon}))\,\mathrm{d}\boldsymbol{\epsilon} & (92) \\ \mathbf{c}(.) = \boldsymbol{l}_{SE(3)}(\mathbf{X}_0, .)^\top & (93) \end{cases}$$

Consequently, the constraint (31) applied on each $\boldsymbol{\Upsilon}_k$ can be used so as to obtain a new inequality:

$$\mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\boldsymbol{l}_{SE(3)}\left(\mathbf{X}_0, \widehat{\mathbf{X}}_0\right)\left(\boldsymbol{l}_{SE(3)}(\mathbf{X}_0, \widehat{\mathbf{X}}_0\right)^\top\right) \succeq \boldsymbol{\Gamma}^\top \mathbf{R}_{\mathbf{s}_K}^{-1}\boldsymbol{\Gamma}, \quad (94)$$

with

$$\begin{cases} \boldsymbol{\Gamma} = [\boldsymbol{\Gamma}(\boldsymbol{\Upsilon}_1), \ldots, \boldsymbol{\Gamma}(\boldsymbol{\Upsilon}_K)]^\top & (95) \\ \mathbf{R}_{\mathbf{s}_K} = \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\mathbf{q}(\mathbf{Z})\,\mathbf{q}(\mathbf{Z})^\top\right) & (96) \\ \mathbf{q}(\mathbf{Z}) = [q(\boldsymbol{\Upsilon}_1,\mathbf{Z}), \ldots, q(\boldsymbol{\Upsilon}_K,\mathbf{Z})]^\top & (97) \end{cases}$$

The right-hand term identifies itself as the IMSB obtained with $L \to \infty$.

4.3.2 Limit of the IMSB when $K \to \infty$

According to the equation (2) [41], the IMSB (30) is achieved if $\mathbf{R}_{\mathbf{s}_K}^{-1}\boldsymbol{\Gamma} = \boldsymbol{\Lambda}\,\mathbf{B}$ with:

$$\boldsymbol{\Lambda} = \mathrm{diag}\Big[|\boldsymbol{\phi}_{SE(3)}(-\tilde{\boldsymbol{\tau}}_1)|\boldsymbol{l}_S(\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2)^\top; \ldots, |\boldsymbol{\phi}_{SE(3)}(-\tilde{\boldsymbol{\tau}}_K)|\,\boldsymbol{l}_S(\boldsymbol{\Upsilon}_K, \boldsymbol{\Upsilon}_{K+1})^\top\Big] \in \mathbb{R}^{K\times 6\,K}, \quad (98)$$

and $\mathbf{B} \in \mathbb{R}^{6\,K\times 6}$. $\tilde{\boldsymbol{\tau}}_k$ is defined so that $\boldsymbol{\Upsilon}_k = \mathrm{Exp}^\wedge_{SE(3)}(\tilde{\boldsymbol{\tau}}_k)\ \ \forall k \in \{1, \ldots, K\}$. It ensues that the i^{th} column of $\boldsymbol{\Gamma}^\top \mathbf{R}_{\mathbf{s}_K}^{-1}\boldsymbol{\Gamma}$ is,

$$\left[\boldsymbol{\Gamma}^\top \mathbf{R}_{\mathbf{s}_K}^{-1}\boldsymbol{\Gamma}\right]_i = \sum_{k=1}^{K} \boldsymbol{\Gamma}(\boldsymbol{\Upsilon}_k)^\top\,\mathbf{b}_i^\top |\boldsymbol{\phi}_{SE(3)}(-\tilde{\boldsymbol{\tau}}_k)|\,\boldsymbol{l}_S(\boldsymbol{\Upsilon}_k, \boldsymbol{\Upsilon}_{k+1}) \quad (99)$$

where $\mathbf{b}_i$ is the i^{th} column of $\mathbf{B}$.

According to the structure of S, we know that,

$$\boldsymbol{l}_S(\boldsymbol{\Upsilon}_k, \boldsymbol{\Upsilon}_{k+1}) = \begin{bmatrix} \mathbf{0}_{3\times 1} \\ \boldsymbol{\tau}_{K+1} - \boldsymbol{\tau}_K \end{bmatrix}, \quad (100)$$

and the previous equation can be developed as,

$$\left[\boldsymbol{\Gamma}^\top \mathbf{R}_{\mathbf{s}_K}^{-1}\boldsymbol{\Gamma}\right]_i = \sum_{k=1}^{K} \boldsymbol{\Gamma}(\mathrm{Exp}^\wedge_{SE(3)}(\tilde{\boldsymbol{\tau}}_k))^\top |\boldsymbol{\phi}_{SE(3)}(-\tilde{\boldsymbol{\tau}}_k)|\tilde{\mathbf{b}}_{k,i}^\top\,(\boldsymbol{\tau}_{k+1} - \boldsymbol{\tau}_k), \quad (101)$$

where $\tilde{\mathbf{b}}_{k,i} = \boldsymbol{\Pi}_k\,\mathbf{b}_i$ and $\boldsymbol{\Pi}_k = [\mathbf{0}_{3\times 6\,k-3}, \mathbf{I}_3\ \mathbf{0}_{3\times 6\,K-6\,k}]$. If the difference $\boldsymbol{\tau}_{k+1} - \boldsymbol{\tau}_k$ is assumed to be constant equal to $\alpha\,\mathbf{e}$ (with $\mathbf{e} = \mathbf{1}_{6\times 1}$) then the sum (101) is a series converging to a Riemannian integral on $\mathbb{R}^6$:

$$\lim_{K\to\infty}\left[\boldsymbol{\Gamma}^\top \mathbf{R}_{\mathbf{s}_K}{}^{-1}\boldsymbol{\Gamma}\right]_i = \alpha\int_{\mathbb{R}^6} \boldsymbol{\Gamma}(\mathrm{Exp}^\wedge_{SE(3)}(\tilde{\boldsymbol{\tau}}))^\top\,\tilde{\mathbf{b}}_i(\mathrm{Exp}^\wedge_{SE(3)}(\tilde{\boldsymbol{\tau}}))^\top \times |\boldsymbol{\phi}_{SE(3)}(-\tilde{\boldsymbol{\tau}})|\,\mathbf{e}\,d\boldsymbol{\tau}. \quad (102)$$

According to (12),

$$\lim_{K\to\infty}\left[\boldsymbol{\Gamma}^\top \mathbf{R}_{s_K}^{-1}\boldsymbol{\Gamma}\right]_i = \alpha\int_{SE(3)} \boldsymbol{\Gamma}(\boldsymbol{\Upsilon})^\top\,\tilde{\mathbf{b}}_i^\top(\boldsymbol{\Upsilon})\,\mathbf{e}\,\lambda_{SE(3)}(d\,\boldsymbol{\Upsilon}). \quad (103)$$

By using (11) and (92), one obtains:

$$\lim_{K\to\infty}\left[\boldsymbol{\Gamma}^\top \mathbf{R}_{s_K}^{-1}\boldsymbol{\Gamma}\right]_i = \alpha\int_{SE(3)}\int_{SE(3)} h(\boldsymbol{\Upsilon},\mathbf{X})\,\mathbf{c}(\mathbf{X})^\top\tilde{\mathbf{b}}_i^\top(\boldsymbol{\Upsilon})\,\mathbf{e}\,\lambda_{SE(3)}(d\,\boldsymbol{\Upsilon})\lambda_{SE(3)}(d\,\mathbf{X}). \quad (104)$$

Consider, as in [40], that $\mathbf{H}_K = [\mathbf{h}_1, \ldots, \mathbf{h}_K]$ tends to $\mathbf{I}$ when $L \to \infty$. In the Euclidean case, it means the function $h(\cdot)$ corresponds to the Dirac distribution. This distribution can be generalized for parameters lying on $SE(3)$ under the following form [15],

$$h(\boldsymbol{\Upsilon},\mathbf{X}) = \boldsymbol{\delta}(\boldsymbol{\Upsilon},\mathbf{X}) \triangleq \prod_{i=1}^{6}\delta\left(\left[\boldsymbol{l}_{SE(3)}(\boldsymbol{\Upsilon},\mathbf{X})\right]_i\right) \quad (105)$$

where δ is the Dirac distribution on $\mathbb{R}$. Consequently,

$$\lim_{K\to\infty}\left[\boldsymbol{\Gamma}^\top\mathbf{R}_{\mathbf{s}_K}^{-1}\boldsymbol{\Gamma}\right]_i = \alpha\int_{SE(3)} \mathbf{c}(\mathbf{X})^\top\tilde{\mathbf{b}}_i(\mathbf{X})^\top\mathbf{e}\lambda_{SE(3)}(d\mathbf{X}). \quad (106)$$

5 Validation

In this section, we propose to numerically validate the proposed intrinsic bounds in the case of Gaussian observation model (59) for both $SO(3)$ and $SE(3)$ LGs. The bounds are compared to the corresponding IMSE.

5.1 Implementation details

5.1.1 Implementation of the intrinsic bounds

- $N = 3$ 3D points are assumed with $\mathbf{p}_1 = [1,2,2]^\top$, $\mathbf{p}_2 = [3,4,5]^\top$ and $\mathbf{p}_3 = [0.1, 0.2, 2]^\top$.
- Σ is assumed equal to $\sigma^2 \mathbf{I}_3$.
- To compute the ICRB, the only quantities we need are the generators $\{\mathbf{G}_s\}_{s=1}^p$ for $SO(3)$ and $SE(3)$. They are provided in App. A.
- Concerning the IMSB, we have to define the test points. We assume that $\forall l \in \{1, \ldots, L-1\}$, every test point $\mathbf{X}^{(l)}$ is written as:

$$\mathbf{X}^{(l)} = \mathbf{X}_0 \operatorname{Exp}_G^\wedge(\boldsymbol{\delta}_l) \tag{107}$$

  where $\boldsymbol{\delta}_l$ is a zero-mean Gaussian vector with covariance matrix $\sigma_l^2 \mathbf{I}$. This generative model allows to browse the whole group, especially when σ_l is sufficiently high.

5.1.2 Implementation of the IMSE

The IMSE expression is given by the trace of equation (19). As the latter is not tractable, it can be approximated by Monte-Carlo simulations:

$$\frac{1}{N_{mc}} \sum_{t=1}^{N_{mc}} \left\| \boldsymbol{l}_G(\mathbf{X}_0, \left(\widehat{\mathbf{X}}_0\right)_t) \right\|^2 \tag{108}$$

where N_{mc} is the number of realizations and $\left(\widehat{\mathbf{X}}_0\right)_t$ the t^{th} realization of the estimator. The estimator $\left(\widehat{\mathbf{X}}_0\right)_t$ is computed by searching the likelihood maximum of $p(\mathbf{z}|\mathbf{X}_0)$. It amounts to find the minima of the criterion $\sum_{n=1}^{N} \|\mathbf{z}_n - \mathbf{\Pi}\, \mathbf{X}_0 \mathbf{\Pi}'\, \mathbf{p}_n\|^2_{\mathbf{\Sigma}}$. To obtain a sufficiently accurate estimator, a Gauss-Newton algorithm on LGs is used [42]. As the criterion is potentially not convex, it is not sure that the algorithm converges to a global minimum. Nevertheless, convergence properties have been verified numerically in several works [43][44][45].

5.2 Case of $SO(3)$

We first simulate the proposed bounds with the model applied to $SO(3)$. We assume an unknown rotation matrix $\mathbf{X}_0 = \operatorname{Exp}^\wedge_{SO(3)}([0.1, 0.1, 0.1])$. In Fig. 5.2, we first observe the consistency of the proposed bounds. Indeed, as well as the IMSE, they have values increasing whatever the value of σ^2, and both are lower bounds on such value. Secondly, we remark that for low σ^2 values, the IMSB and ICRB have a weak gap and are close to the IMSE. The greater is the value of σ^2, the closer the two bounds becomes. Thus, for a model on $SO(3)$, the IMSB and the ICRB results are similar.

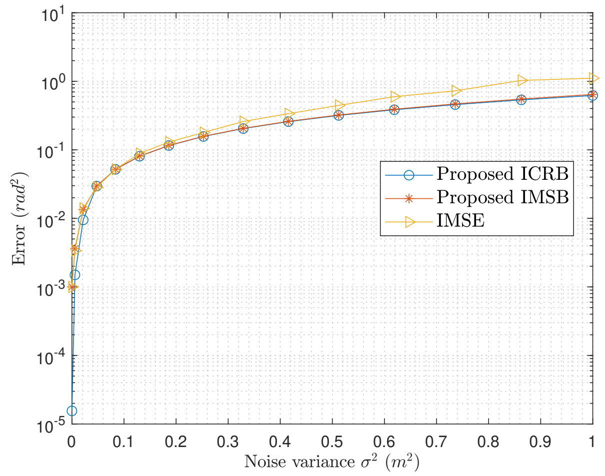

Fig. 4. Evolution of the ICRB and IMSB superimposing to the IMSE for $SO(3)$.

5.3 Case of $SE(3)$

Now we are interested in the model on $SE(3)$. We assume $\mathbf{X}_0 = \operatorname{Exp}^\wedge_{SE(3)}([0.1, 0.1, 0.1, 1, 1, 1])$. Fig. 5 shows the IMSB, ICRB and the corresponding IMSE for different values of the noise observation model variance.

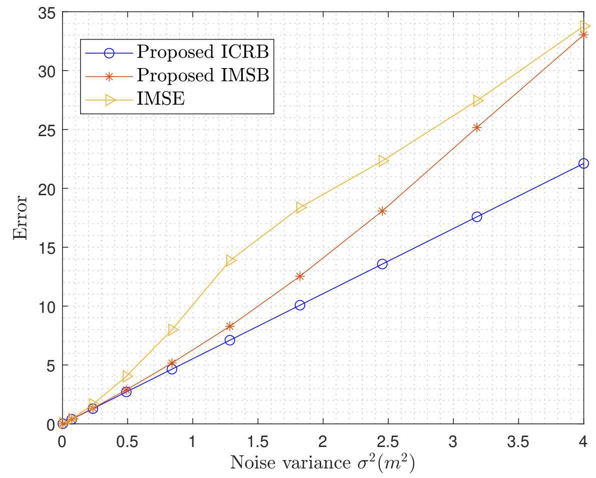

Fig. 5. Evolution of the ICRB and IMSB superimposing to the IMSE for $SE(3)$

As previously seen for the $SO(3)$ case, we observe that the computations of the bounds are coherent. Indeed, both bounds stay inferior to the IMSE independently of the σ^2 value, that is, they are lower bounding the IMSE. Moreover, we remark that for small values of σ^2, both IMSB and ICRB are almost equivalent. In contrast to the $SO(3)$ case, when σ^2 becomes large, they tend to drift apart. Particularly, the IMSB becomes tighter and getting closer to the IMSE. Concerning the ICRB, we observe that it tends to overestimate the theoretical performance of the model for high noise variances. Consequently, for the model on $SE(3)$ with large noise, the IMSB is the preferred option as it provides a tighter bound.

In order to analyze more precisely the relation between both bounds, we have also studied the influence of the test point increment $\boldsymbol{\delta}_l$. Fig. 5.3 shows the ICRB and IMSB for different values of the increment test point standard deviation. We observe that a high value of $\sigma_l (\simeq 10^{-1})$ provides two bounds with an important gap. This observation confirms the methodology because the ICRB is built by tending σ_l to zero.

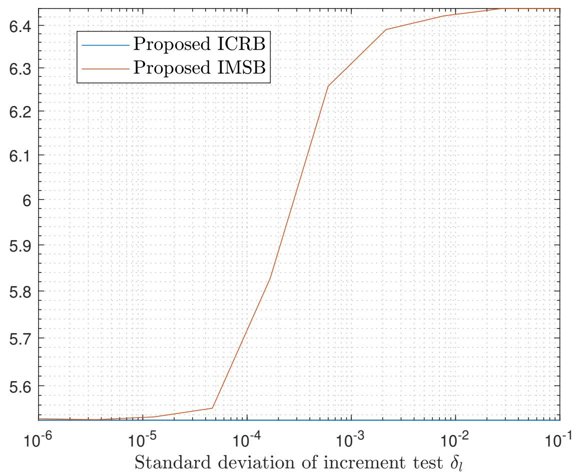

Fig. 6. Evolution of the ICRB and IMSB for different values of increment $\boldsymbol{\delta}_l$ with $\sigma^2 = 1$.

6 Conclusions

In this article, we developed a new IMSB on LGs and provided a new formalization of the ICRB. The starting point was to adapt to the LG case the existing Euclidean formalism where the bounds of interest can be derived from the general BB. The three IBB, IMSB and ICRB expressions were obtained by using LG tools. Especially, we obtained an ICRB similar to the one proposed in [14], but the ICRB in this article generalizes existing results in the literature. Closed-form expressions of these two bounds were obtained for a Gaussian observation model and parameters on $SO(3)$ and $SE(3)$. These expressions were validated through numerical simulations, as expected it was found that the IMSB may be a tighter bound for some problems. A future work is to extend the proposed bounds to the case where unknown parameters belong jointly to a LG and a discrete parameter space. This specificity is particularly encountered in multi-antenna carrier phase-based precise positioning systems, where both the receiver attitude and a set of integer ambiguities must be estimated.

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References

[1] P.-A. Absil, R. Mahony, and R. Sepulchre. *Optimization algorithm on matrix manifolds.* Princeton, 2008.

[2] N. Boumal. *Optimization and estimation on manifolds.* PhD thesis, Louvain university, 2014.

[3] X. Pennec. Intrinsic statistics on Riemannian manifolds : basic tools for geometric measurements. *J. Math. Imaging and Vision*, 25:127–155, 2006.

[4] H. Snoussi and A. Mohammad-Djafari. Particle filtering on manifolds. In *AIP Conference Proceeding*, 2006.

[5] P. Zanini, M. Congedo, C. Jutten, S. Said, and Y. Berthoumieu. Parameters estimate of Riemannian Gaussian distribution in the manifold of covariance matrices. In *IEEE Sensor Array and Multichannel Signal Processing Workshop*, 2016.

[6] D.H.S. Maithripala and J. M. Berg. An intrinsic PID controller for mechanical systems on lie groups. *Automatica*, 54:189–200, 2015.

[7] Brossard, M. and Bonnabel, S. and Barrau, A. Invariant Kalman Filtering for Visual Inertial SLAM. In *21st International Conference on Information Fusion*, 21st International Conference on Information Fusion, Cambrige, United Kingdom, Jul 2018. University of Cambridge.

[8] J. Ćesić, I. Marković, M. Bukal, and I. Petrović. Extended information filter on matrix Lie groups. *Automatica*, 82:226–234, 2017.

[9] Jianwei Wu. A Completed Adaptive De-mixing Algorithm on Stiefel Manifold for ICA, September 2009.

[10] S.T. Smith. Covariance, subspace, and intrinsic Cramér-Rao bounds. *IEEE Transactions On Signal Processing*, 53:1610–1630, 2005.

[11] N. Boumal. On Intrinsic Cramér-Rao bounds for Riemannian submanifolds and quotient manifolds. *IEEE Transactions On Signal Processing*, 61:1809–1821, 2013.

[12] A. Breloy, G. Ginolhac, A. Renaux, and F. Bouchard. Intrinsic Cramér-Rao Bounds for Scatter and Shape Matrices Estimation in CES Distributions. *IEEE Sig. Proc. Letters*, 26(2):262–266, 2019.

[13] V. Solo and G. S. Chirikjian. On the Cramér-Rao bound in Riemannian manifolds with application to $SO(3)$. In *IEEE International Conference on Decision and Control*, 2020.

[14] S. Bonnabel and A. Barrau. An intrinsic Cramér-Rao bound on Lie groups. In *International Conference on Networked Geometric Science of Information*, pages 664–672, 2015.

[15] G. S. Chirikjian. Information theory on Lie groups and mobile robotics applications. In *IEEE International Conference on Robotics and Automation*, 2010.

[16] N. Boumal, A. Singer, P.-A. Absil, and V.D. Blondelu. Cramér–Rao bounds for synchronization of rotations. *Information and Inference: A Journal of the IMA*, 3:1–39, 2014.

[17] G. Bourmaud, R. Megret, A. Giremus, and Y. Berthoumieu. Global motion estimation from relative measurements using iterated extended Kalman filter on matrix Lie groups. In *IEEE Inter. Conf. on Image Proc*, Proc. of ICIP 2014, Paris, France, October 2014.

[18] G. Bourmaud. Online Variational Bayesian Motion Averaging. In *ECCV*, Amsterdam, Netherlands, 2016.

[19] N. Bergman. On the Cramér-Rao bound for terrain-aided navigation, 1997.

[20] E. W. Barankin. Locally best unbiased estimates. *Annals of Mathematical Statistics*, 20:477–501, 1949.

[21] R. McAulay and E. M. Hofstetter. Barankin Bounds on parameter estimation. *IEEE Trans. Inf. Theory*, 17(6):669–676, 1971.

[22] F. E. Glave. A new look at the Barankin lower bound. *IEEE Trans. Inf. Theory*, 18:349–356, May 1972.

[23] N. Kbayer, J. Galy, E. Chaumette, F. Vincent, A. Renaux, and P. Larzabal. On lower bounds for nonstandard deterministic estimation. *IEEE Trans. Signal Process.*, 65(6):1538–1553, 2017.

[24] R. McAulay and L. P. Seidman. A useful form of the Barankin lower bound and its application to PPM threshold analysis. *IEEE Trans. Inf. Theory*, 15(2):273–279, Mar. 1969.

[25] D. Medina, J. Vilà-Valls, E. Chaumette, F. Vincent, and P. Closas. Cramér-Rao Bound for a Mixture of Real- and Integer-Valued Parameter Vectors and Its Application to the Linear Regression Model. *Signal Processing*, 179:107792, February 2021.

[26] Jon M. Selig. Geometrical methods in robotics. In *Monographs in Computer Science*, 1996.

[27] J. Gallier and J. Quaintance. *Differential Geometry and Lie Groups A Computational Perspective.* Springer, 2020.

[28] J. Faraut. *Analysis on Lie Groups: An Introduction.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2008.

[29] G. Chirikjian. *Stochastic Models, Information Theory, and Lie Groups, Volume 2*, pages 129–154. Jan 2012.

[30] A. Barrau and S. Bonnabel. Intrinsic filtering on lie groups with applications to attitude estimation. *IEEE Transactions on Automatic Control*, 60(2):436–449, 2015.

[31] T. Menni, E. Chaumette, P. Larzabal, and J. P. Barbot. New results on Deterministic Cramér-Rao bounds for real and complex parameters. *IEEE Trans. Signal Process.*, 60(3):1032–1049, 2012.

[32] S. Bonnabel and A. Barrau. An intrinsic cramér-rao bound on SO(3) for dynamic attitude filtering. *2015 54th IEEE Conference on Decision and Control (CDC)*, pages 2158–2163, 2015.

[33] G. Wahba. A least squares estimate of satellite attitude. *SIAM Review*, 7(3):409–409, 1965.

[34] X. Lu. A review of solutions for perspective-n-point problem in camera pose estimation. *Journal of Physics: Conference Series*, 1087:052009, Sep 2018.

[35] Forsyth D. A and Ponce J. *Computer Vision, A Modern Approach.* Practice Hall, New York, USA, 2003.

[36] T. D. Barfoot and P. T. Furgale. Associating uncertainty with three-dimensional poses for use in estimation problems. *IEEE Transactions on Robotics*, 30:679–693, 2014.

[37] V. S. Varadarajan. *Lie groups, Lie algebras, and their representations.* Springer-Verlag, 1984.

[38] Slepian D. Estimation of signal parameters in the presence of noise,. *Trans. IRE Prof. Group Inform. Theory PG IT-3*, 74:68–69, 1954.

[39] W. J. Bangs. *Array processing with generalized beamformers.* PhD thesis, Yale Univ., New Haven, CT,, Novembre 1971.

[40] E. Chaumette, A. Renaux, and P. Larzabal. Lower bounds on the mean square error derived from mixture of linear and non-linear transformations of the unbiasness definition. In *2009 IEEE International Conference on Acoustics, Speech and Signal Processing*, pages 3045–3048, 2009.

[41] E. Chaumette, A. Renaux, and P. Larzabal. New trends in deterministic lower bounds and snr threshold estimation: From derivable bounds to conjectural bounds. In *2010 IEEE Sensor Array and Multichannel Signal Processing Workshop*, pages 121–124, 2010.

[42] G. Bourmaud, R. Mégret, A. Giremus, and Y. Berthoumieu. From Intrinsic Optimization to Iterated Extended Kalman Filtering on Lie Groups. *Journal of Mathematical Imaging and Vision*, 55(3), July 2016.

[43] B. Xie, S. Dai, and F. Liu. A Lie Group-Based Iterative Algorithm Framework for Numerically Solving Forward Kinematics of Gough–Stewart Platform. *Mathematics*, 9(7), 2021.

[44] P. Lee and J. Moore. Gauss-Newton-on-manifold for pose estimation. *Journal of Industrial and Management Optimization*, 1, 11 2005.

[45] U. Helmke, K. Hueper, P. Lee, and J. Moore. Essential Matrix Estimation Using Gauss-Newton Iterations on a Manifold. *International Journal of Computer Vision*, 74:117–136, 05 2007.

A Expressions of $SO(3)$ and $SE(3)$ generators

A.1 Case of $SO(3)$

The generators are skew-symmetric elementary matrix such as $\forall \mathbf{w} \in \mathbb{R}^3$:

$$[\mathbf{w}]_\times = \sum_{i=1}^{3} w_i \mathbf{G}_i \tag{A.1}$$

Thereby:

$$\mathbf{G}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{G}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \mathbf{G}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A.2}$$

A.2 Case of $SE(3)$

Let the vector $\mathbf{w} \in \mathbb{R}^6$ which can be divided $[\mathbf{v}, \mathbf{u}]$. The generators of $SE(3)$ are defined as a basis of the Lie algebra such as:

$$\begin{bmatrix} [\mathbf{v}]_\times & \mathbf{u} \\ \mathbf{0} & 0 \end{bmatrix} = \sum_{i=1}^{6} w_i \mathbf{G}_i \tag{A.3}$$

Then, we gather:

$$\mathbf{G}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{G}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{G}_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.4}$$

$$\mathbf{G}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G}_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.5}$$

B Demonstration of equation (65)

Let us consider the term:

$$\begin{aligned} c^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) &= -\mathbf{z}_n^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\Pi}\, \mathbf{X}_0 \ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1) \ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2) \ \boldsymbol{\Pi}'\, \mathbf{p}_n \\ &+ \frac{1}{2} \mathbf{p}_n^\top \boldsymbol{\Pi}'^\top \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)^\top \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)^\top \mathbf{A} \\ &\times \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1) \ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2) \ \boldsymbol{\Pi}'\, \mathbf{p}_n \end{aligned} \tag{B.1}$$

As this therm is equal to the Hessian of $\log p(\mathbf{z}|\mathbf{X}_0)$, we have by definition :

$$\mathbf{P}_{ICRB} = \left(\mathbb{E}_{p(\mathbf{z}|\mathbf{X}_0)} \left(- \sum_{i=1}^{N} \frac{\partial^2 c^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)}{\partial \boldsymbol{\delta}_1 \, \partial \boldsymbol{\delta}_2} \right) \Bigg|_{\boldsymbol{\delta}_1 = \mathbf{0}, \boldsymbol{\delta}_2 = \mathbf{0}} \right)^{-1} \tag{B.2}$$

By linearity of the expectation operator, the ICRB is also written:

$$\mathbf{P}_{ICRB} = \left(\sum_{i=1}^{N} \mathbb{E}_{p(\mathbf{z}|\mathbf{X}_0)} \left(- \frac{\partial^2 c^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)}{\partial \boldsymbol{\delta}_1 \, \partial \boldsymbol{\delta}_2} \right) \Bigg|_{\boldsymbol{\delta}_1 = \mathbf{0}, \boldsymbol{\delta}_2 = \mathbf{0}} \right)^{-1} \tag{B.3}$$

What's more, we can decompose $c^{(i)}$ such as:

$$c^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) = e^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) + f^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) \tag{B.4}$$

with:

$$e^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) = -\mathbf{z}_n^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\Pi}\, \mathbf{X}_0 \ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1) \ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2) \ \boldsymbol{\Pi}'\, \mathbf{p}_n \tag{B.5}$$

$$\begin{aligned} f^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) = \frac{1}{2} \mathbf{p}_n^\top \boldsymbol{\Pi}'^\top \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)^\top \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)^\top \mathbf{A} \times \\ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1) \ \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2) \ \boldsymbol{\Pi}'\, \mathbf{p}_n \end{aligned} \tag{B.6}$$

Thus, we get:

$$\mathbf{P}_{ICRB} = \Bigg(\underbrace{\sum_{i=1}^{N} \mathbf{E}^{(i)} + \sum_{i=1}^{N} \mathbf{F}^{(i)}}_{\mathcal{J}} \Bigg)^{-1} \tag{B.7}$$

with:

$$\mathbf{E}^{(i)} = \mathbb{E}_{p(\mathbf{z}|\mathbf{X}_0)} \left(- \frac{\partial^2 e^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)}{\partial \boldsymbol{\delta}_1 \, \partial \boldsymbol{\delta}_2} \right) \tag{B.8}$$

$$\mathbf{F}^{(i)} = \mathbb{E}_{p(\mathbf{z}|\mathbf{X}_0)} \left(- \frac{\partial^2 f^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)}{\partial \boldsymbol{\delta}_1 \, \partial \boldsymbol{\delta}_2} \right) \tag{B.9}$$

Consequently, the ICRB can be gathered by computing $\mathbf{E}^{(i)}$ and $\mathbf{F}^{(i)}$.

B.1 Computation of $\mathbf{E}^{(i)}$*:*

$$\begin{aligned} \frac{\partial e^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)}{\partial \boldsymbol{\delta}_1 \, \partial \boldsymbol{\delta}_1} &= -\mathbf{z}_n^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\Pi}\, \mathbf{X}_0 \ \frac{\partial \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_1)}{\partial \boldsymbol{\delta}_1} \Bigg|_{\boldsymbol{\delta}_1 = \mathbf{0}} \\ &\frac{\partial \mathrm{Exp}_G^\wedge(\boldsymbol{\delta}_2)}{\partial \boldsymbol{\delta}_2} \Bigg|_{\boldsymbol{\delta}_2 = \mathbf{0}} \boldsymbol{\Pi}'\, \mathbf{p}_n \end{aligned} \tag{B.10}$$

First, let us note that

$$\frac{\partial \mathrm{Exp}_G^\wedge(\boldsymbol{\delta})}{\partial \boldsymbol{\delta}_l} \Bigg|_{\boldsymbol{\delta} = \mathbf{0}} = \mathbf{G}_l$$

where $\boldsymbol{\delta}_l$ is the component l of $\boldsymbol{\delta}$. Consequently, we have:

$$\frac{\partial e^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)}{\partial \left(\boldsymbol{\delta}_1\right)_k \, \left(\boldsymbol{\delta}_2\right)_l} = -2\, \mathbf{z}_n^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\Pi}\, \mathbf{X}_0\, \mathbf{G}_k\, \mathbf{G}_l \boldsymbol{\Pi}'\, \mathbf{p}_n \tag{B.11}$$

It implies that:

$$\begin{aligned} \left[\mathbf{E}^{(i)}\right]_{k,l} &= \mathbb{E}_{p(\mathbf{z}|\mathbf{X}_0)} \left(-\mathbf{z}_n^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\Pi}\, \mathbf{X}_0\, \mathbf{G}_k\, \mathbf{G}_l \boldsymbol{\Pi}'\, \mathbf{p}_n \right) \\ &= -\mathbf{p}_n^\top \left(\boldsymbol{\Pi}'\right)^\top \ \mathbf{A}\, \mathbf{G}_k\, \mathbf{G}_l\, \boldsymbol{\Pi}'\, \mathbf{p}_n \end{aligned} \tag{B.12}$$

B.2 Computation of $\mathbf{F}^{(i)}$

To derive an expression of $\mathbf{F}_i$, we have to compute the quantity $\dfrac{\partial f^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)}{\partial \left(\boldsymbol{\delta}_1\right)_k \, \partial \left(\boldsymbol{\delta}_2\right)_l}$. We proceed in two steps:

- First, we derive the quantity $\dfrac{\partial f^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)}{\partial \left(\boldsymbol{\delta}_1\right)_k}$,

- Then, the gathered expression is differentiated according to $(\boldsymbol{\delta}_2)_l$ and computed for $\boldsymbol{\delta}_1 = \boldsymbol{\delta}_2 = \mathbf{0}$.

We show, by classical classic derivation rules, that:

$$\frac{\partial f^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)}{\partial\left(\boldsymbol{\delta}_1\right)_k} = f_1(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) + f_2(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) \tag{B.13}$$

with:

$$f_1(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) = \frac{1}{2}\,\mathbf{p}_n^\top \boldsymbol{\Pi}'^\top\ \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_2\right)\ ^\top\ \frac{\partial \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_1\right)}{\partial\left(\boldsymbol{\delta}_1\right)_k}\ ^\top\ \mathbf{A}\times$$
$$\mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_1\right)\ \ \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_2\right)\ \ \boldsymbol{\Pi}'\,\mathbf{p}_n \tag{B.14}$$

$$f_2(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) = \frac{1}{2}\,\mathbf{p}_n^\top \boldsymbol{\Pi}'^\top \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_2\right)\ ^\top\ \ \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_1\right)\ \ \mathbf{A}\times$$
$$\frac{\partial \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_1\right)}{\partial\left(\boldsymbol{\delta}_1\right)_k}\ \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_2\right)\ \ \boldsymbol{\Pi}'\,\mathbf{p}_n \tag{B.15}$$

Then, by deriving according to $(\boldsymbol{\delta}_2)_l$, we yield to:

$$\frac{\partial f_1(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)}{\partial\left(\boldsymbol{\delta}_2\right)_l} = g_1(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) + g_2(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) \tag{B.16}$$

$$\frac{\partial f_2(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)}{\partial\left(\boldsymbol{\delta}_2\right)_l} = g_3(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) + g_4(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) \tag{B.17}$$

with:

$$g_1(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) = \frac{1}{2}\,\mathbf{p}_n^\top \boldsymbol{\Pi}'^\top \frac{\partial \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_2\right)}{\partial\left(\boldsymbol{\delta}_2\right)_l}\ ^\top\ \frac{\partial \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_1\right)}{\partial\left(\boldsymbol{\delta}_1\right)_k}\ ^\top\ \mathbf{A}\times$$
$$\mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_1\right)\ \ \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_2\right)\ \ \boldsymbol{\Pi}'\,\mathbf{p}_n \tag{B.18}$$

$$g_2(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) = \frac{1}{2}\,\mathbf{p}_n^\top \boldsymbol{\Pi}'^\top \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_2\right)\ ^\top\ \frac{\partial \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_1\right)}{\partial\left(\boldsymbol{\delta}_1\right)_k}\ ^\top\ \mathbf{A}$$
$$\mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_1\right)\ \ \frac{\partial \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_2\right)}{\partial\left(\boldsymbol{\delta}_2\right)_l}\ \boldsymbol{\Pi}'\,\mathbf{p}_n \tag{B.19}$$

$$g_3(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) = \frac{1}{2}\,\mathbf{p}_n^\top \boldsymbol{\Pi}'^\top \frac{\partial \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_2\right)}{\partial\left(\boldsymbol{\delta}_2\right)_l}\ ^\top\ \ \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_1\right)\ \ \mathbf{A}\times$$
$$\frac{\partial \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_1\right)}{\partial\left(\boldsymbol{\delta}_1\right)_k}\ \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_2\right)\ \ \boldsymbol{\Pi}'\,\mathbf{p}_n \tag{B.20}$$

$$g_4(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) = \frac{1}{2}\,\mathbf{p}_n^\top \boldsymbol{\Pi}'^\top\ \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_2\right)\ ^\top\ \ \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_1\right)\ ^\top\,\mathbf{A}\times$$
$$\frac{\partial \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_1\right)}{\partial\left(\boldsymbol{\delta}_1\right)_k}\ \frac{\partial \mathrm{Exp}_G^\wedge\left(\boldsymbol{\delta}_2\right)}{\partial\left(\boldsymbol{\delta}_2\right)_l}\ \boldsymbol{\Pi}'\,\mathbf{p}_n \tag{B.21}$$

Thus, we have :

$$\left.\frac{\partial^2 f^{(i)}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)}{\partial\left(\boldsymbol{\delta}_1\right)_k\,\partial\left(\boldsymbol{\delta}_2\right)_l}\right|_{\boldsymbol{\delta}_1=\mathbf{0},\boldsymbol{\delta}_2=\mathbf{0}} = g_1(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) + g_2(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) +$$
$$g_3(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) + g_4(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)\big|_{\boldsymbol{\delta}_1=\mathbf{0},\boldsymbol{\delta}_2=\mathbf{0}} \tag{B.22}$$

The expression of $\mathbf{F}^{(i)}$ is obtained by taking the expectation of (B.22). Then :

$$\left[\mathbf{F}^{(i)}\right]_{k,l} = a_1 + a_2 + a_3 + a_4 \tag{B.23}$$

with:

$$a_1 = \frac{1}{2}\,\mathbf{p}_n^\top \boldsymbol{\Pi}'^\top \mathbf{G}_l^\top\ \mathbf{G}_k^\top \mathbf{A}\,\boldsymbol{\Pi}'\,\mathbf{p}_n \tag{B.24}$$

$$a_2 = \frac{1}{2}\,\mathbf{p}_n^\top \boldsymbol{\Pi}'^\top\ \mathbf{G}_k^\top\ \mathbf{A}\,\mathbf{G}_l\,\boldsymbol{\Pi}'\,\mathbf{p}_n \tag{B.25}$$

$$a_3 = \frac{1}{2}\,\mathbf{p}_n^\top \boldsymbol{\Pi}'^\top \mathbf{G}_l^\top\ \mathbf{A}\,\mathbf{G}_k\,\boldsymbol{\Pi}'\,\mathbf{p}_n \tag{B.26}$$

$$a_4 = \frac{1}{2}\,\mathbf{p}_n^\top \boldsymbol{\Pi}'^\top\ \mathbf{A}\,\mathbf{G}_k\,\mathbf{G}_l\,\boldsymbol{\Pi}'\,\mathbf{p}_n \tag{B.27}$$